Vector Bundles over Multipullback Quantum Complex Projective Spaces*

Albert Jeu-Liang Sheu†

Department of Mathematics, University of Kansas, Lawrence, KS 66045, U. S. A.

e-mail: asheu@ku.edu

Abstract

We work on the classification of isomorphism classes of finitely generated projective modules over the C*-algebras $C\left(\mathbb{P}^n\left(\mathcal{T}\right)\right)$ and $C\left(\mathbb{S}_H^{2n+1}\right)$ of the quantum complex projective spaces $\mathbb{P}^n\left(\mathcal{T}\right)$ and the quantum spheres $\mathbb{S}_H^{2n+1}$, and the quantum line bundles L_k over $\mathbb{P}^n\left(\mathcal{T}\right)$, studied by Hajac and collaborators. Motivated by the groupoid approach of Curto, Muhly, and Renault to the study of C*-algebraic structure, we analyze $C\left(\mathbb{P}^n\left(\mathcal{T}\right)\right)$, $C\left(\mathbb{S}_H^{2n+1}\right)$, and L_k in the context of groupoid C*-algebras, and then apply Rieffel's stable rank results to show that all finitely generated projective

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modules over $C\left(\mathbb{S}_H^{2n+1}\right)$ of rank higher than $\left\lfloor \frac{n}{2} \right\rfloor + 3$ are free modules. Furthermore, besides identifying a large portion of the positive cone of the K_0-group of $C\left(\mathbb{P}^n\left(\mathcal{T}\right)\right)$, we also explicitly identify L_k with concrete representative elementary projections over $C\left(\mathbb{P}^n\left(\mathcal{T}\right)\right)$.

Keywords: multipullback quantum projective space; multipullback quantum sphere; quantum line bundle; finitely generated projective module; cancellation problem; Toeplitz algebra of polydisk; groupoid C*-algebra; stable rank; noncommutative vector bundle

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1 Introduction

Since the concept of noncommutative geometry first popularized by Connes [5], many interesting examples of a C*-algebra $\mathcal{A}$ viewed as the algebra $C(X_q)$ of continuous functions on a virtual quantum space X_q have been constructed with a topological or geometrical motivation, and analyzed in comparison with their classical counterpart. For example, quantum odd-dimensional spheres and associated complex projective spaces have been introduced and studied by Soibelman, Vaksman, Meyer, and others [32, 14] as $\mathbb{S}_q^{2n+1}$ and $\mathbb{C}P_q^n$ via a quantum universal enveloping algebra approach, and by Hajac and his collaborators including Baum, Kaygun, Matthes, Nest, Pask, Sims, Szymański, Zieliński, and others [2, 10, 9, 12] as $\mathbb{S}_H^{2n+1}$ and $\mathbb{P}^n(\mathcal{T})$ via a multi-pullback and Toeplitz algebra approach. Actually $\mathbb{S}_H^{2n+1}$ is the untwisted special case of the more general version of θ-twisted spheres $\mathbb{S}_{H,\theta}^{2n+1}$ introduced in [12].

Motivated by Swan's work [30], the concept of a noncommutative vector bundle E_q over a quantum space X_q can be reformulated as a finitely generated projective (left) module $\Gamma(E_q)$ over $C(X_q)$. Based on the strong connection approach to quantum principal bundles [8] for compact quantum groups [34, 35], Hajac and his collaborators introduced quantum line bundles L_k of degree k over $\mathbb{P}^n(\mathcal{T})$ as some rank-one projective modules realized as spectral subspaces $C\left(\mathbb{S}_H^{2n+1}\right)_k$ of $C\left(\mathbb{S}_H^{2n+1}\right)$ under a $U(1)$-action [12]. Besides having the K_0-group of $C(\mathbb{P}^n(\mathcal{T}))$ computed, they found that L_k is not stably free unless $k=0$, extending earlier results for the case of $n=1$ [10, 11].

It has always been an interesting but challenging task to classify finitely generated projective modules over an algebra up to isomorphism, which goes beyond their classification up to stable isomorphism by K_0-group and appears in the form of so-called cancellation problem. Classically it is known that the cancellation law holds for complex vector bundles of rank no less than $\frac{d}{2}$ over a d-dimensional CW-complex, which implies that all complex vector bundles over $\mathbb{S}^{2n+1}$ of rank $n+1$ or above are trivial.

The study of such classification problem for C*-algebras was popularized by Rieffel [21, 22] who introduced useful versions of stable ranks for C*-algebras to facilitate the analysis involved. Some successes have been achieved for certain quantum algebras [22, 23, 25, 1, 19]. In particular, Peterka showed that all finitely generated projective modules over the θ-deformed 3-spheres S^3_θ are free, and constructed all those over S^4_θ up to isomorphism [19]. With more effort, the result of Bach [1] on the cancellation law for $\mathbb{S}^{2n+1}_q$ and $\mathbb{C}P^n_q$ can be strengthened to a complete classification of finitely generated projective modules over them, which we will address elsewhere.

With the K_0-group of $C(\mathbb{P}^n(\mathcal{T}))$ known [12], it is natural to try to classify finitely generated projective modules over $C(\mathbb{P}^n(\mathcal{T}))$ and identify the line bundles L_k among them. In [29], a complete solution was obtained for the special case of $n = 1$.

In this paper, we use the powerful groupoid approach to C*-algebras initiated by Renault [20] and popularized by Curto, Muhly, and Renault [6, 15] to study multi-variable Toeplitz C*-algebras $\mathcal{T}^{\otimes n}$, quantum spheres $C\left(\mathbb{S}^{2n+1}_H\right)$, and quantum complex projective spaces $C(\mathbb{P}^n(\mathcal{T}))$. Utilizing results on stable ranks of C*-algebras obtained by Rieffel [21], we analyze finitely generated projective modules over $\mathcal{T}^{\otimes n+1}$ and $C\left(\mathbb{S}^{2n+1}_H\right)$, and get those of rank higher than $\left\lfloor \frac{n}{2} \right\rfloor + 3$ and also a large class of "standard" modules classified up to isomorphism. Furthermore, besides identifying a large portion of the positive cone of the K_0-group $K_0\left(C\left(\mathbb{P}^n\left(\mathcal{T}\right)\right)\right)$, we explicitly identify the quantum line bundles L_k with concrete representative elementary projections.

On the other hand, there are still a lot of questions to be further investigated, e.g. whether the cancellation law holds for low-ranked finitely generated projective modules, and whether the more general case of θ-twisted multipullback quantum sphere $\mathbb{S}_{H,\theta}^{2n+1}$ brings in new phenomena. Finally it is of interest to note the recent work of Farsi, Hajac, Maszczyk, and Zieliński [7] on $K_0\left(C\left(\mathbb{P}^2\left(\mathcal{T}\right)\right)\right)$, identifying its free generators arising from Milnor modules as sums of L_k, which can then be expressed in terms of elementary projections by our result.

2 Notations

Taking the groupoid approach to C*-algebras initiated by Renault [20] and popularized by the work of Curto, Muhly, and Renault [6, 15], we give a description of the C*-algebras $C\left(\mathbb{S}_H^{2n-1}\right)$ and $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ of [12] as some concrete groupoid C*-algebras. We refer to [20, 15] for the concepts and theory of groupoid C*-algebras used freely in the following discussion.

By abuse of notation, for any C*-algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, we denote the C*-algebra homomorphism $M_k\left(\phi\right) : M_k\left(\mathcal{A}\right) \to M_k\left(\mathcal{B}\right)$ for $k \in \mathbb{N} \equiv \{1, 2, 3, ...\}$ also by ϕ. We use $\mathcal{A}^\times$ to denote the set of all invertible elements of an algebra $\mathcal{A}$, and use $\mathcal{A}^+$ to denote the minimal unitization of $\mathcal{A}$. For any topological group G, we use G^0 to denote the identity component of G, i.e. the connected component that contains the identity element of G.

We denote by $M_\infty(\mathcal{A})$ the direct limit (or the union as sets) of the increasing sequence of matrix algebras $M_n(\mathcal{A})$ over $\mathcal{A}$ with the canonical inclusion $M_n(\mathcal{A}) \subset M_{n+1}(\mathcal{A})$ identifying $x \in M_n(\mathcal{A})$ with $x \boxplus 0 \in M_{n+1}(\mathcal{A})$ for any algebra $\mathcal{A}$, where $\boxplus$ denotes the standard diagonal concatenation (sum) of two matrices. So the size of an element in $M_\infty(\mathcal{A})$ can be taken arbitrarily large. We also use $GL_\infty(\mathcal{A})$ to denote the direct limit of the general linear groups $GL_n(\mathcal{A})$ over a unital C*-algebra $\mathcal{A}$ with $GL_n(\mathcal{A})$ embedded in $GL_{n+1}(\mathcal{A})$ by identifying $x \in GL_n(\mathcal{A})$ with $x \boxplus 1 \in GL_{n+1}(\mathcal{A})$.

By an idempotent P over a unital C*-algebra $\mathcal{A}$, we mean an element $P \in M_\infty(\mathcal{A})$ with $P^2 = P$, and a self-adjoint idempotent in $M_\infty(\mathcal{A})$ is called a projection over $\mathcal{A}$. Two idempotents $P, Q \in M_\infty(\mathcal{A})$ are called equivalent, denoted as $P \sim Q$, if there exists $U \in GL_\infty(\mathcal{A})$ such that $UPU^{-1} = Q$. Each idempotent $P \in M_n(\mathcal{A})$ over $\mathcal{A}$ defines a finitely generated left projective module $E := \mathcal{A}^n P$ over $\mathcal{A}$ where elements of $\mathcal{A}^n$ are viewed as row vectors. The mapping $P \mapsto \mathcal{A}^n P$ induces a bijective correspondence between the equivalence classes of idempotents over $\mathcal{A}$ and the isomorphism classes of finitely generated left projective modules over $\mathcal{A}$ [3]. From now on, by a module over $\mathcal{A}$, we mean a left $\mathcal{A}$-module, unless otherwise specified.

Two finitely generated projective modules E, F over $\mathcal{A}$ are called stably isomorphic if they become isomorphic after being augmented by the same finitely generated free $\mathcal{A}$-module, i.e. $E \oplus \mathcal{A}^k \cong F \oplus \mathcal{A}^k$ for some $k \geq 0$. Correspondingly, two idempotents P and Q are called stably equivalent if $P \boxplus I_k$ and $Q \boxplus I_k$ are equivalent for some identity matrix I_k. The

K_0-group $K_0(\mathcal{A})$ classifies idempotents over $\mathcal{A}$ up to stable equivalence. The classification of idempotents over a C*-algebra up to equivalence, appearing as the so-called cancellation problem, was popularized by Rieffel's pioneering work [21, 22] and is in general an interesting but difficult question.

The set of all equivalence classes of idempotents over a C*-algebra $\mathcal{A}$ is an abelian monoid $\mathfrak{P}(\mathcal{A})$ with its binary operation provided by the diagonal sum $\boxplus$. The image of the canonical homomorphism from $\mathfrak{P}(\mathcal{A})$ into $K_0(\mathcal{A})$ is the so-called positive cone of $K_0(\mathcal{A})$.

Furthermore, it is well-known [3] that in the above descriptions of $\mathfrak{P}(\mathcal{A})$ and $K_0(\mathcal{A})$, one can restrict to the self-adjoint idempotents, called projections over $\mathcal{A}$, and their unitary equivalence classes, which faithfully represent the elements of $\mathfrak{P}(\mathcal{A})$ and $K_0(\mathcal{A})$.

In this paper, we use freely the basic techniques and manipulations for K-theory found in [3, 31].

For a Hilbert space $\mathcal{H}$, we denote the C*-algebra consisting of all compact linear operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$, or simply by $\mathcal{K}$ if $\mathcal{H}$ is the essentially unique separable infinite-dimensional Hilbert space.

In the following, we use the notations $\mathbb{Z}_{\geq k} := \{n \in \mathbb{Z} | n \geq k\}$ and $\mathbb{Z}_{\geq} := \mathbb{Z}_{\geq 0}$. In particular, $\mathbb{N} = \mathbb{Z}_{\geq 1}$. We use I to denote the identity operator canonically contained in $\mathcal{K}^+ \subset \mathcal{B}(\ell^2(\mathbb{Z}_{\geq}))$, and

$$P_m := \sum_{i=1}^{m} e_{ii} \in M_m(\mathbb{C}) \subset \mathcal{K}$$

to denote the standard $m \times m$ identity matrix in $M_m(\mathbb{C}) \subset \mathcal{K}$ for any integer $m \geq 0$ (with

$M_0(\mathbb{C}) = 0$ and $P_0 = 0$ understood). We also use the notation

$$P_{-m} := I - P_m \in \mathcal{K}^+$$

for integers $m > 0$, and take symbolically $P_{-0} \equiv I - P_0 = I \neq P_0$.

3 Quantum spaces as groupoid C*-algebras

Let $\mathfrak{T}_n := \left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_\geq}^n}$ with $n \geq 1$ be the transformation group groupoid $\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n$ restricted to the positive "cone" $\overline{\mathbb{Z}_\geq}^n$ where $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{+\infty\}$ containing $\mathbb{Z}_\geq \equiv \{n \in \mathbb{Z} | n \geq 0\}$ carries the standard topology, and $\mathbb{Z}^n$ acts on $\overline{\mathbb{Z}}^n$ componentwise in the canonical way. From the groupoid isomorphism

$$\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_\geq}^n} \cong \times^n \left(\left(\mathbb{Z} \ltimes \overline{\mathbb{Z}}\right)\big|_{\overline{\mathbb{Z}_\geq}}\right)$$

and the well-known C*-algebra isomorphism $C^*\left(\left(\mathbb{Z} \ltimes \overline{\mathbb{Z}}\right)\big|_{\overline{\mathbb{Z}_\geq}}\right) \cong \mathcal{T}$ for the Toeplitz C*-algebra $\mathcal{T}$, we get the groupoid C*-algebra

$$C^*(\mathfrak{T}_n) \equiv C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_\geq}^n}\right) \cong \mathcal{T}^{\otimes n} \equiv \otimes^n \mathcal{T}.$$

We consider two important nontrivial invariant open subsets of the unit space $\overline{\mathbb{Z}_\geq}^n$ of $\mathfrak{T}_n$, namely, $\mathbb{Z}_\geq^n$ the smallest one and $\overline{\mathbb{Z}_\geq}^n \setminus \{\infty^n\}$ the largest one, where $\infty^n := (\infty, ..., \infty) \in \overline{\mathbb{Z}_\geq}^n$. By the theory of groupoid C*-algebras developed in Renault's book [20], they give rise to two short exact sequences of C*-algebras

$$0 \to C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\mathbb{Z}_\geq^n}\right) \cong \mathcal{K}\left(\ell^2\left(\mathbb{Z}_\geq^n\right)\right) \to C^*(\mathfrak{T}_n) \equiv \mathcal{T}^{\otimes n} \to C^*(\mathfrak{G}_n) \to 0$$

with $\mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^n\right)\right) \cong \otimes^n \mathcal{K} \equiv \mathcal{K}^{\otimes n}$ where

$$\mathfrak{G}_n := \left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_{\geq}}^n \setminus \mathbb{Z}_{\geq}^n}$$

is $\mathfrak{T}_n$ restricted to the "limit boundary" of its unit space, and

$$0 \to C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_{\geq}}^n \setminus \{\infty^n\}}\right) \to C^*\left(\mathfrak{T}_n\right) \equiv \mathcal{T}^{\otimes n} \stackrel{\sigma_n}{\to} C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\{\infty^n\}}\right) \cong C^*\left(\mathbb{Z}^n\right) \cong C\left(\mathbb{T}^n\right) \to 0$$

where the quotient map σ_n extends the notion of the well-known symbol map σ on $\mathcal{T}$ in the case of $n = 1$.

Note that the open invariant set $\mathbb{Z}_{\geq}^n$ being dense in the unit space $\overline{\mathbb{Z}_{\geq}}^n$ of $\mathfrak{T}_n$ induces a faithful representation π_n of $C^*\left(\mathfrak{T}_n\right)$ on $\ell^2\left(\mathbb{Z}_{\geq}^n\right)$ that realizes the groupoid C*-algebra $C^*\left(\mathfrak{T}_n\right)$ and its closed ideal $C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\mathbb{Z}_{\geq}^n}\right)$ respectively as a C*-subalgebra of $\mathcal{B}\left(\ell^2\left(\mathbb{Z}_{\geq}^n\right)\right)$ and the closed ideal $\mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^n\right)\right)$ consisting of all compact operators on $\ell^2\left(\mathbb{Z}_{\geq}^n\right)$.

In this paper, we freely identify elements of $C^*\left(\mathfrak{T}_n\right) \equiv \mathcal{T}^{\otimes n}$ with operators on $\ell^2\left(\mathbb{Z}_{\geq}^n\right)$ via the faithful representation π_n and use these two conceptually different notions interchangeably.

In [12], Hajac, Nest, Pask, Sims, and Zieliński defined the (untwisted) *multipullback* or *Heegaard* quantum odd-dimensional sphere S_H^{2n-1} as the quantum space of the multipullback C*-algebra [18] determined by homomorphisms of the form $\mathrm{id}^{\otimes j} \otimes \sigma \otimes \mathrm{id}^{\otimes n-j-1}$ from $\mathcal{T}^{\otimes i} \otimes C\left(\mathbb{T}\right) \otimes \mathcal{T}^{\otimes n-i-1}$ with $i \neq j$ to some $\mathcal{T}^{\otimes m} \otimes C\left(\mathbb{T}\right) \otimes \mathcal{T}^{\otimes k} \otimes C\left(\mathbb{T}\right) \otimes \mathcal{T}^{\otimes n-m-k-2}$. (Actually more general θ-twisted quantum spheres $S_{H,\theta}^{2n-1}$ are studied there.) They showed that

$$C\left(\mathbb{S}_H^{2n-1}\right) \cong \left(\otimes^n \mathcal{T}\right) / \left(\otimes^n \mathcal{K}\right)$$

and hence we have

$$C\left(\mathbb{S}_H^{2n-1}\right) \cong C^*\left(\mathfrak{G}_n\right)$$

identified as a groupoid C*-algebra.

With the ideal $C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_\geq}^n \setminus \{\infty^n\}}\right)$ containing the ideal $C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\mathbb{Z}_\geq^n}\right)$, the quotient map σ_n induces a well-defined quotient map τ_n in the short exact sequence

$$0 \to C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_\geq}^n \setminus \left(\mathbb{Z}_\geq^n \cup \{\infty^n\}\right)}\right) \to C\left(\mathbb{S}_H^{2n-1}\right) \cong \left(\otimes^n \mathcal{T}\right) / \left(\otimes^n \mathcal{K}\right) \xrightarrow{\tau_n} C\left(\mathbb{T}^n\right) \to 0.$$

4 Stable ranks of quantum spaces

In his seminal paper [21], Rieffel introduced and popularized the notions of topological stable rank $\operatorname{tsr}(\mathcal{A})$ and connected stable rank $\operatorname{csr}(\mathcal{A})$ of a C*-algebra $\mathcal{A}$, which are useful tools in the study of cancellation problems for finitely generated projective modules. Later, Herman and Vaserstein [13] showed that for C*-algebras $\mathcal{A}$, Rieffel's topological stable rank coincides with the Bass stable rank used in algebraic K-theory. So we will denote $\operatorname{tsr}(\mathcal{A})$ simply as $\operatorname{sr}(\mathcal{A})$ in our discussion.

In this section, we review an estimate of the stable ranks of the Toeplitz algebras $\mathcal{T}^{\otimes n}$ and quantum spheres $C\left(\mathbb{S}_H^{2n-1}\right)$, which will be used in our study of their finitely generated projective modules. For the case of $n = 1$, it is known [21] that $\operatorname{sr}(\mathcal{T}) = \operatorname{csr}(C(\mathbb{T})) = 2$.

As an illustration of the groupoid approach to C*-algebras, we first establish some composition sequence structure for $\mathcal{T}^{\otimes n}$ and $C\left(\mathbb{S}_H^{2n-1}\right)$, which leads to an easy estimate of their stable ranks.

Proposition 1. There is a finite composition sequence of closed ideals

$$\mathcal{T}^{\otimes n} \equiv C^*(\mathfrak{T}_n) \equiv \mathcal{I}_n \rhd \mathcal{I}_{n-1} \rhd \cdots \rhd \mathcal{I}_1 \rhd \mathcal{I}_0 \rhd \mathcal{I}_{-1} \equiv \{0\}$$

such that $\mathcal{T}^{\otimes n}/\mathcal{I}_0 \cong C\left(\mathbb{S}_H^{2n-1}\right)$, and for $0 \leq j \leq n$,

$$\mathcal{I}_j/\mathcal{I}_{j-1} \cong \oplus^{\frac{n!}{j!(n-j)!}} \left(\mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n-j}\right)\right) \otimes C\left(\mathbb{T}^j\right)\right),$$

where $\mathbb{T}^0$ and $\mathbb{Z}_{\geq}^0$ denote a singleton.

Proof. For $0 \leq j \leq n$, let X_j be the set consisting of $z \in \overline{\mathbb{Z}_{\geq}}^n$ with exactly j of the components $z_1, z_2, ..., z_n$ being equal to ∞, and hence $X_n = \{\infty^n\}$. Then the sets

$$Y_j := X_0 \sqcup X_1 \sqcup \cdots \sqcup X_j$$

are open invariant subsets of the unit space $\overline{\mathbb{Z}_{\geq}}^n$ of $\mathfrak{T}_n$ with

$$\mathbb{Z}_{\geq}^n = Y_0 \subset Y_1 \subset \cdots \subset Y_n = \overline{\mathbb{Z}_{\geq}}^n$$

which determines an increasing chain of closed ideals $\mathcal{I}_0 \lhd \mathcal{I}_1 \lhd \cdots \lhd \mathcal{I}_n$ of $C^*(\mathfrak{T}_n)$ defined by

$$\mathcal{I}_j := C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{Y_j}\right) \equiv C^*\left(\mathfrak{T}_n|_{Y_j}\right).$$

Note that $Y_j \backslash Y_{j-1} = X_j$ with $Y_{-1} := \emptyset$ is a disjoint union of $\frac{n!}{j!(n-j)!}$ copies of $\mathbb{Z}_{\geq}^{n-j} \times \{\infty^j\}$ each of which is gotten from one of the $\frac{n!}{j!(n-j)!}$ possible selections of exactly j of the n components of $\overline{\mathbb{Z}_{\geq}}^n$. With each such copy of $\mathbb{Z}_{\geq}^{n-j} \times \{\infty^j\}$ clearly a closed invariant subset of $Y_j \backslash Y_{j-1}$, these $\frac{n!}{j!(n-j)!}$ copies of $\mathbb{Z}_{\geq}^{n-j} \times \{\infty^j\}$ are open invariant subsets of $Y_j \backslash Y_{j-1}$, and hence

$$C^*\left(\mathfrak{T}_n|_{Y_j \backslash Y_{j-1}}\right) = \oplus^{\frac{n!}{j!(n-j)!}} C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\big|_{\mathbb{Z}_{\geq}^{n-j} \times \{\infty^j\}}\right)$$

$$= \oplus^{\frac{n!}{j!(n-j)!}} C^* \left(\left(\left(\mathbb{Z}^{n-j} \ltimes \mathbb{Z}^{n-j} \right) \big|_{\mathbb{Z}_{\geq}^{n-j}} \right) \times \mathbb{Z}^j \right) = \oplus^{\frac{n!}{j!(n-j)!}} \left(\mathcal{K} \left(\ell^2 \left(\mathbb{Z}_{\geq}^{n-j} \right) \right) \otimes C \left(\mathbb{T}^j \right) \right).$$

Thus with $\mathcal{I}_j = C^* \left(\mathfrak{T}_n|_{Y_j} \right)$ and $\mathcal{I}_{j-1} = C^* \left(\mathfrak{T}_n|_{Y_{j-1}} \right)$, we get

$$\mathcal{I}_j / \mathcal{I}_{j-1} \cong C^* \left(\mathfrak{T}_n|_{Y_j \setminus Y_{j-1}} \right) \cong \oplus^{\frac{n!}{j!(n-j)!}} \left(\mathcal{K} \left(\ell^2 \left(\mathbb{Z}_{\geq}^{n-j} \right) \right) \otimes C \left(\mathbb{T}^j \right) \right).$$

□

Corollary 1. There is a finite composition sequence of closed ideals

$$C \left(\mathbb{S}_H^{2n-1} \right) \equiv C^* \left(\mathfrak{G}_n \right) \equiv \mathcal{J}_n \rhd \mathcal{J}_{n-1} \rhd \cdots \rhd \mathcal{J}_1 \rhd \mathcal{J}_0 \equiv \{0\}$$

such that for $1 \leq j \leq n$,

$$\mathcal{J}_j / \mathcal{J}_{j-1} \cong \oplus^{\frac{n!}{j!(n-j)!}} \left(\mathcal{K} \left(\ell^2 \left(\mathbb{Z}_{\geq}^{n-j} \right) \right) \otimes C \left(\mathbb{T}^j \right) \right).$$

Proof. With $\mathcal{I}_0 = \mathcal{K} \left(\ell^2 \left(\mathbb{Z}_{\geq}^{n} \right) \right)$ and hence $C^* \left(\mathfrak{T}_n \right) / \mathcal{I}_0 \cong C \left(\mathbb{S}_H^{2n-1} \right)$, we simply take $\mathcal{J}_j := \mathcal{I}_j / \mathcal{I}_0$. □

The above composition sequences lead to the straightforward estimates

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \operatorname{sr} \left(C \left(\mathbb{S}_H^{2n-1} \right) \right) \leq \operatorname{sr} \left(\mathcal{T}^{\otimes n} \right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

and

$$\operatorname{csr} \left(\mathcal{T}^{\otimes n} \right) \leq \operatorname{csr} \left(C \left(\mathbb{S}_H^{2n-1} \right) \right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

for all $n \geq 1$, based on the general rules established in [21] that (i) $\operatorname{sr} \left(\mathcal{A} \otimes \mathcal{K} \right) = \min \left\{ 2, \operatorname{sr} \left(\mathcal{A} \right) \right\}$, (ii) for any closed ideal $\mathcal{I}$ of a C*-algebra $\mathcal{A}$,

$$\max \left\{ \operatorname{sr} \left(\mathcal{A}/\mathcal{I} \right), \operatorname{sr} \left(\mathcal{I} \right) \right\} \leq \operatorname{sr} \left(\mathcal{A} \right) \leq \max \left\{ \operatorname{sr} \left(\mathcal{A}/\mathcal{I} \right), \operatorname{sr} \left(\mathcal{I} \right), \operatorname{csr} \left(\mathcal{A}/\mathcal{I} \right) \right\},$$

and (iii) $\mathrm{sr}\,(C\,(X)) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ for any n-dimensional CW-complex X, and the rule [25, 16, 17] that for any closed ideal $\mathcal{I}$ of a C*-algebra $\mathcal{A}$, (iv) $\mathrm{csr}\,(\mathcal{A} \otimes \mathcal{K}) \leq 2$ (with $\mathrm{csr}\,(\mathcal{K}) = 1$) and (v)

$$\mathrm{csr}\,(\mathcal{A}) \leq \max\{\mathrm{csr}\,(\mathcal{A}/\mathcal{I}), \mathrm{csr}\,(\mathcal{I})\}.$$

Indeed, for $n > 1$, applying (i)-(ii) and (iv)-(v) to the short exact sequences

$$0 \to \mathcal{I}_{j-1} \to \mathcal{I}_j \to \mathcal{I}_j/\mathcal{I}_{j-1} \cong \oplus^{\frac{n!}{j!(n-j)!}} \left(\mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n-j}\right)\right) \otimes C\left(\mathbb{T}^j\right)\right) \to 0$$

inductively for j increasing from 1 to $n-1$, starting with the exact sequence

$$0 \to \mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n}\right)\right) \cong \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_1/\mathcal{I}_0 \cong \oplus^{n} \left(\mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n-1}\right)\right) \otimes C\left(\mathbb{T}\right)\right) \to 0$$

for $j = 1$, we get $\mathrm{csr}\,(\mathcal{I}_j), \mathrm{sr}\,(\mathcal{I}_j) \leq 2$ for all $1 \leq j \leq n-1$. In particular, $\mathrm{csr}\,(\mathcal{I}_{n-1}), \mathrm{sr}\,(\mathcal{I}_{n-1}) \leq 2$, which is also valid for $n = 1$ since $\mathcal{I}_0 \cong \mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n}\right)\right)$. Then with $\mathrm{csr}\,(C\,(\mathbb{T}^n)) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$ by homotopy theory [33], we get $\mathrm{csr}\,(\mathcal{T}^{\otimes n}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$ and

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \mathrm{sr}\left(\mathcal{T}^{\otimes n}\right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

by further applying (ii)-(iii) and (v) to the short exact sequence

$$0 \to \mathcal{I}_{n-1} \to \mathcal{I}_n \equiv \mathcal{T}^{\otimes n} \to \mathcal{I}_n/\mathcal{I}_{n-1} \cong C\,(\mathbb{T}^n) \to 0.$$

Similar argument yields $\mathrm{csr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$ and $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \mathrm{sr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$, with the inequality $\mathrm{sr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \leq \mathrm{sr}\,(\mathcal{T}^{\otimes n})$ obviously valid by (ii). Also $\mathrm{csr}\,(\mathcal{T}^{\otimes n}) \leq \mathrm{csr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ by (iv)-(v).

Such an estimate determining $\mathrm{sr}\,(\mathcal{T}^{\otimes n})$ sharply for even n and up to an error of 1 for odd $n > 1$ as stated above was first obtained by G. Nagy in [16] and then sharpened to the exact value

$$\mathrm{sr}\left(\mathcal{T}^{\otimes n}\right)=\left\lfloor\frac{n}{2}\right\rfloor+1\ \ \text{(and hence }\ \mathrm{sr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)=\left\lfloor\frac{n}{2}\right\rfloor+1\text{)}$$

for general $n>1$ by Nistor in [17] which also gives $\mathrm{csr}\left(\mathcal{T}^{\otimes n}\right)\leq\left\lfloor\frac{n+1}{2}\right\rfloor+1$. We summarize these results as follows.

Proposition 2. For all $n>1$,

$$\mathrm{sr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)=\mathrm{sr}\left(\mathcal{T}^{\otimes n}\right)=\left\lfloor\frac{n}{2}\right\rfloor+1$$

and

$$\mathrm{csr}\left(\mathcal{T}^{\otimes n}\right)\leq\mathrm{csr}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)\leq\left\lfloor\frac{n+1}{2}\right\rfloor+1.$$

Corollary 2. For any $n>1$ and any $k\geq\left\lfloor\frac{n}{2}\right\rfloor+3$, the topological group $GL_k\left(\mathcal{T}^{\otimes n}\right)$ is connected.

Proof. By the Künneth formula [3] for K-groups, we get $K_1\left(\mathcal{T}^{\otimes n}\right)=0$ since $K_1\left(\mathcal{T}\right)=0$ is well known. So by the theorem [21] that $K_1\left(\mathcal{A}\right)\cong GL_k\left(\mathcal{A}\right)/GL_k^0\left(\mathcal{A}\right)$ for any unital C*-algebra $\mathcal{A}$ with $k\geq\mathrm{sr}\left(\mathcal{A}\right)+2$, we get $GL_k\left(\mathcal{T}^{\otimes n}\right)=GL_k^0\left(\mathcal{T}^{\otimes n}\right)$ for any $k\geq\left\lfloor\frac{n}{2}\right\rfloor+3\geq\mathrm{sr}\left(\mathcal{T}^{\otimes n}\right)+2$.

□

Note that the above statement holds for the case of $n=1$, since $GL_k\left(\mathcal{T}\right)$ is connected for all $k\geq 1$ in the case of $n=1$ by the index theory of Toeplitz operators for the unit disk $\mathbb{D}$.

5 Projective modules over $\mathcal{T}^{\otimes n}$

Before proceeding to study finitely generated projective modules over $\mathcal{T}^{\otimes n}$, we now point out a structure of $\mathcal{T}^{\otimes n}$ which facilitates some inductive procedures for the study of such modules.

For all $n \in \mathbb{N}$, the topological groupoid $\mathfrak{T}_n|_{\overline{\mathbb{Z}_\geq}^{n-1}\times\{\infty\}}$ is isomorphic to the product topological groupoid $\mathfrak{T}_{n-1} \times \mathbb{Z}$, while the topological groupoid $\mathfrak{T}_n|_{\overline{\mathbb{Z}_\geq}^{n-1}\times\mathbb{Z}_\geq}$ is isomorphic to the product topological groupoid $\mathfrak{T}_{n-1} \times (\mathbb{Z} \ltimes \mathbb{Z})|_{\mathbb{Z}_\geq}$, where the closed subset $\overline{\mathbb{Z}_\geq}^{n-1} \times \{\infty\}$ and its open complement $\overline{\mathbb{Z}_\geq}^{n-1} \times \mathbb{Z}_\geq$ in the unit space $\overline{\mathbb{Z}_\geq}^{n}$ of $\mathfrak{T}_n$ are invariant. (Here it is understood that when $n-1=0$, the first factor $\overline{\mathbb{Z}_\geq}^{n-1}$ is dropped.) Hence we get the short exact sequence of C*-algebras

$$0 \to C^*\left(\mathfrak{T}_n|_{\overline{\mathbb{Z}_\geq}^{n-1}\times\mathbb{Z}_\geq}\right) \cong \mathcal{T}^{\otimes n-1} \otimes \mathcal{K}\left(\ell^2\left(\mathbb{Z}_\geq\right)\right) \to$$
$$C^*\left(\mathfrak{T}_n\right) \equiv \mathcal{T}^{\otimes n} \stackrel{\kappa_n}{\to} C^*\left(\mathfrak{T}_n|_{\overline{\mathbb{Z}_\geq}^{n-1}\times\{\infty\}}\right) \cong \mathcal{T}^{\otimes n-1} \otimes C\left(\mathbb{T}\right) \to 0$$

with $\mathcal{T}^{\otimes 0} := \mathbb{C}$. Furthermore the quotient maps κ_n for $n \in \mathbb{N}$ resulting from a groupoid restriction satisfy the commuting diagram

$$\begin{array}{ccccc}
M_k\left(\mathcal{T}^{\otimes n}\right) & \stackrel{\kappa_n}{\to} & M_k\left(\mathcal{T}^{\otimes n-1} \otimes C\left(\mathbb{T}\right)\right) & \equiv & M_k\left(\mathcal{T}^{\otimes n-1}\right) \otimes C\left(\mathbb{T}\right) \\
\downarrow_{\sigma_n} & \circlearrowleft & \downarrow_{\sigma_{n-1}\otimes \mathrm{id}} & & \downarrow_{\sigma_{n-1}\otimes \mathrm{id}} \\
M_k\left(C\left(\mathbb{T}^n\right)\right) & \stackrel{\equiv}{\to} & M_k\left(C\left(\mathbb{T}^{n-1}\right) \otimes C\left(\mathbb{T}\right)\right) & \equiv & M_k\left(C\left(\mathbb{T}^{n-1}\right)\right) \otimes C\left(\mathbb{T}\right)
\end{array}$$

where $\equiv$ stands for a canonical isomorphism and $\sigma_0 := \mathrm{id}_{\mathbb{C}}$.

To classify the isomorphism classes of finitely generated projective $\mathcal{T}^{\otimes n}$-modules E or equivalently the equivalence classes of idempotents $P \in M_\infty\left(\mathcal{T}^{\otimes n}\right)$ over $\mathcal{T}^{\otimes n}$, we first define the rank of (the class of) E or P as the classical rank of (the isomorphism class of) the vector bundle corresponding to (the class of) the $C(\mathbb{T}^n)$-module $C(\mathbb{T}^n) \otimes_{\mathcal{T}^{\otimes n}} E$ or the projection $\sigma_n(P)$ over $C(\mathbb{T}^n)$.

The set of equivalence classes of idempotents $P \in M_\infty(\mathcal{T}^{\otimes n})$ equipped with the binary operation $\boxplus$ becomes an abelian graded monoid

$$\mathfrak{P}\left(\mathcal{T}^{\otimes n}\right) = \sqcup_{m=0}^{\infty} \mathfrak{P}_m\left(\mathcal{T}^{\otimes n}\right)$$

where $\mathfrak{P}_m(\mathcal{T}^{\otimes n})$ is the set of all (equivalence classes of) idempotents over $\mathcal{T}^{\otimes n}$ of rank m, and

$$\mathfrak{P}_m\left(\mathcal{T}^{\otimes n}\right) \boxplus \mathfrak{P}_l\left(\mathcal{T}^{\otimes n}\right) \subset \mathfrak{P}_{m+l}\left(\mathcal{T}^{\otimes n}\right)$$

for $m, l \geq 0$. Clearly $\mathfrak{P}_0(\mathcal{T}^{\otimes n})$ is a submonoid of $\mathfrak{P}(\mathcal{T}^{\otimes n})$.

Next we define a submonoid of $\mathfrak{P}(\mathcal{T}^{\otimes n})$ generated by "standard" type of idempotents, which turns out to contain (equivalence classes of) all idempotents of sufficiently high ranks, and then classify its elements.

Note that each permutation Θ on $\{1, 2, ..., n\}$ induces canonically a C*-algebra automorphism, still denoted as Θ by abuse of notation, on $\mathcal{T}^{\otimes n}$ by permuting the indices of the factors in $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \mathcal{T}^{\otimes n}$ for $a_i \in \mathcal{T}$. A permutation Θ on $\{1, 2, ..., n\}$ is called a $(j, n-j)$-shuffle on $\{1, 2, ..., n\}$ if $\Theta(1) < \Theta(2) < \cdots < \Theta(j)$ and $\Theta(j+1) < \Theta(j+2) < \cdots < \Theta(n)$.

Some basic projections over $\mathcal{T}^{\otimes n}$ are given by $\Theta(P_{j,l})$ where

$$P_{j,l} := \boxplus^l \left(\left(\otimes^j I\right) \otimes \left(\otimes^{n-j} P_1\right)\right) \in M_l\left(\mathcal{T}^{\otimes n}\right)$$

for $l \geq 0$ and $0 \leq j \leq n$ (in particular, $P_{n,m} \equiv \boxplus^m (\otimes^n I) \equiv \boxplus^m \tilde{I}$ for the unit $\tilde{I}$ of

$\mathcal{T}^{\otimes n}$), and Θ is (the automorphism defined by) a $(j, n-j)$-shuffle on $\{1, 2, ..., n\}$. Note that $\Theta(P_{j,l}) = \Theta\left(\boxplus^l P_{j,1}\right) = \boxplus^l \Theta(P_{j,1})$,

$$\Theta(P_{j,l}) \boxplus \Theta(P_{j,l'}) \sim \Theta(P_{j,l+l'}),$$

and $(\otimes^j I) \otimes (\otimes^{n-j-1} P_1) \otimes P_l \sim P_{j,l}$ over $\mathcal{T}^{\otimes n}$ since $P_l \sim \boxplus^l P_1$ over $\mathcal{K}^+ \subset \mathcal{T}$. Furthermore

$$\sigma_n(\Theta(P_{j,l})) = \begin{cases} 0, & \text{if } \quad 0 \leq j \leq n-1 \\ \boxplus^l 1, & \text{if } \quad j = n \end{cases},$$

and hence $\Theta(P_{j,l}) \in \mathfrak{P}_0(\mathcal{T}^{\otimes n})$ if $j < n$ and $\Theta(P_{n,l}) = P_{n,l} \in \mathfrak{P}_l(\mathcal{T}^{\otimes n})$, where $1 \in C(\mathbb{T}^n)$ is the constant function 1 on $\mathbb{T}^n$. So the set $\mathfrak{P}'_0(\mathcal{T}^{\otimes n}) \subset \mathfrak{P}_0(\mathcal{T}^{\otimes n})$ consisting of (the equivalence classes of) all possible $\boxplus$-sums of $\Theta(P_{j,l})$ with $l \geq 0$ and Θ a $(j, n-j)$-shuffle on $\{1, 2, ..., n\}$ for $0 \leq j \leq n-1$ is a submonoid of $\mathfrak{P}_0(\mathcal{T}^{\otimes n})$. For $m \geq 1$, we define a singleton

$$\mathfrak{P}'_m\left(\mathcal{T}^{\otimes n}\right) := \left\{P_{n,m} \equiv \boxplus^m \tilde{I}\right\} \subset \mathfrak{P}_m\left(\mathcal{T}^{\otimes n}\right)$$

where $\tilde{I}$ denotes the identity element of $\mathcal{T}^{\otimes n}$. Clearly $\sqcup_{m=1}^{\infty} \mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ is also a submonoid of $\mathfrak{P}(\mathcal{T}^{\otimes n})$.

We define a partial ordering $\prec$ on the collection

$$\Omega := \{(j, \Theta) : \ 0 \leq j \leq n \text{ and } \Theta \text{ is a } (j, n-j)\text{-shuffle}\}$$

by the condition that $(j', \Theta') \prec (j, \Theta)$ if and only if $\Theta(\{1, 2, ..., j\}) \supsetneqq \Theta'(\{1, 2, ..., j'\})$ (and hence $j > j'$). Here $\{1, 2, ..., 0\} \equiv \emptyset$ is understood. Note that $\mathrm{id}_{\{1,2,...,n\}}$ is a $(j, n-j)$-shuffle for every j, and $\left(n, \mathrm{id}_{\{1,2,...,n\}}\right)$ is the greatest element while $\left(0, \mathrm{id}_{\{1,2,...,n\}}\right)$ is the smallest element in Ω with respect to $\prec$.

Proposition 3. $\mathfrak{P}'(\mathcal{T}^{\otimes n}) = \sqcup_{m=0}^{\infty} \mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ is a graded submonoid of $\mathfrak{P}(\mathcal{T}^{\otimes n})$ and its monoid structure is explicitly determined by that for any $l, l' > 0$ and any $(j', \Theta') \prec (j, \Theta)$ in Ω,

$$\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l}).$$

Proof. Note that since $\mathfrak{P}'_0(\mathcal{T}^{\otimes n})$ and $\boxplus_{m=1}^{\infty} \mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ are submonoids of $\mathfrak{P}(\mathcal{T}^{\otimes n})$, the set $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ is a submonoid if $\Theta(P_{n,m}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{n,m})$ holds for all $m > 0$ and all $\Theta'(P_{j',l'})$ with $j' \leq n-1$. Since $\left(n, \mathrm{id}_{\{1,2,\ldots,n\}}\right)$ is the greatest element in Ω, it remains to show that $\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l})$ for $n \geq j > j' \geq 0$ with $\Theta(\{1,2,...,j\}) \supset \Theta'(\{1,2,...,j'\})$ and $l, l' > 0$.

Note that for $\Theta(\{1,2,...,j\}) \supset \Theta'(\{1,2,...,j'\})$, there exists a permutation Θ'' (not necessarily a shuffle) on $\{1,2,...,n\}$ such that $\Theta''(\Theta(P_{j,l})) = P_{j,l}$ and $\Theta''(\Theta'(P_{j',l'})) = P_{j',l'}$. (In fact, one can find a permutation Θ'' such that $\Theta''\Theta$ fixes each of $j+1,..,n$, and $\Theta''\Theta'$ is each of $1,2,...,j'$.) So it suffices to prove that

$$P_{j,l} \boxplus P_{j',l'} \sim P_{j,l}$$

whenever $j > j'$ and $l, l' > 0$. Furthermore since $P_{j,l} = \boxplus^l P_{j,1}$, we only need to show that $P_{j,1} \boxplus P_{j',1} \sim P_{j,1}$ for $j > j'$.

Note that $U(P_1 \boxplus I)U^* = 0 \boxplus I$ in $M_2(\mathcal{T})$ for the unitary

$$U := e_{11} \otimes \mathcal{S}^* + e_{22} \otimes \mathcal{S} + e_{21} \otimes e_{11} \in M_2(\mathbb{C}) \otimes \mathcal{T} \equiv M_2(\mathcal{T})$$

where $\mathcal{S} \in \mathcal{T}$ is the (forward) unilateral shift on $\ell^2(\mathbb{Z}_{\geq})$. So

$$P_{j,1} \boxplus P_{j-1,1} = \left(\left(\otimes^j I\right) \otimes \left(\otimes^{n-j} P_1\right)\right) \boxplus \left(\left(\otimes^{j-1} I\right) \otimes \left(\otimes^{n-j+1} P_1\right)\right)$$

$$= \left(\otimes^{j-1} I\right) \otimes (I \boxplus P_1) \otimes \left(\otimes^{n-j} P_1\right) \sim \left(\otimes^{j-1} I\right) \otimes I \otimes \left(\otimes^{n-j} P_1\right) = P_{j,1}.$$

Thus by iteration of this result, we can "expand" $P_{j,1}$ to get for any $0 \leq k < j$,

$$P_{j,1} \sim P_{j,1} \boxplus P_{j-1,1} \boxplus \cdots \boxplus P_{k,1},$$

and hence

$$P_{j,1} \boxplus P_{j',1} \sim P_{j,1} \boxplus P_{j-1,1} \boxplus \cdots \boxplus P_{j'+1,1} \boxplus P_{j',1} \sim P_{j,1}.$$

□

For each $(j, \Theta) \in \Omega$, let $X_\Theta \subset \overline{\mathbb{Z}_{\geq}}^n$ be the invariant closed subset of the unit space of $\mathfrak{T}_n$ consisting of $z \in \overline{\mathbb{Z}_{\geq}}^n$ with $z_k = \infty$ for all $k \in \Theta(\{1, 2, ..., j\})$, and let

$$\sigma_{(j,\Theta)} : C^*(\mathfrak{T}_n) \to C^*\left(\mathfrak{T}_n|_{X_\Theta}\right) \cong C\left(\mathbb{T}^j\right) \otimes \mathcal{T}^{\otimes n-j} \subset C\left(\mathbb{T}^j\right) \otimes \mathcal{B}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n-j}\right)\right)$$

be the canonical quotient map, where the isomorphism implicitly involves a rearrangement of factors by the inverse permutation Θ^{-1}. Here as before, $\mathbb{T}^0$ is a singleton. Defining $\rho_{(j,\Theta)}(P)$ for an idempotent P over $C^*(\mathfrak{T}_n)$ as the rank of the projection operator $\sigma_{(j,\Theta)}(P)(t) \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n-j}\right)\right)$ for any $t \in \mathbb{T}^j$, which depends only on the equivalence class of P, we get a well-defined monoid homomorphism

$$\rho_{(j,\Theta)} : \left(\mathfrak{P}\left(\mathcal{T}^{\otimes n}\right), \boxplus\right) \to (\mathbb{Z}_{\geq} \cup \{\infty\}, +).$$

A (finite) $\boxplus$-sum of (the equivalence classes of) projections $\Theta(P_{j,l})$ indexed by some $(j, \Theta) \in \Omega$ that are mutually unrelated by $\prec$ with $l \equiv l_{(j,\Theta)} > 0$ depending on (j, Θ) is called a reduced $\boxplus$-sum of standard projections over $\mathcal{T}^{\otimes n}$. It is understood that an "empty" $\boxplus$-sum represents the zero projection and is a reduced $\boxplus$-sum. Two reduced $\boxplus$-sums are called different when they have different sets of (mutually $\prec$-unrelated) indices $(j,\Theta)\in\Omega$ or have different weight functions l of (j,Θ). We are going to show that different reduced $\boxplus$-sums are inequivalent projections. Clearly each projection $\Theta(P_{j,l})$ with $(j,\Theta)\in\Omega$ and $l>0$ is a reduced $\boxplus$-sum.

Theorem 1. The submonoid $\mathfrak{P}'(\mathcal{T}^{\otimes n})=\sqcup_{m=0}^{\infty}\mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ of $\mathfrak{P}(\mathcal{T}^{\otimes n})$ consists exactly of reduced $\boxplus$-sums of standard projections over $\mathcal{T}^{\otimes n}$, and different reduced $\boxplus$-sums are mutually inequivalent projections. Furthermore the monoid homomorphism

$$\rho : P\in\mathfrak{P}'\left(\mathcal{T}^{\otimes n}\right)\mapsto\prod_{(j,\Theta)\in\Omega}\rho_{(j,\Theta)}(P)\in\prod_{(j,\Theta)\in\Omega}\overline{\mathbb{Z}_{\geq}}$$

is injective, with $\rho_{(j,\Theta)}(\Theta(P_{j,l}))=l\in\mathbb{N}$.

Proof. By definition, $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ consists of $\boxplus$-sums of (the equivalence classes of) projections $\Theta(P_{j,l})$ with $(j,\Theta)\in\Omega$ and $l>0$. Since $\Theta(P_{j,l})+\Theta(P_{j,l'})\sim\Theta(P_{j,l+l'})$, we only need to consider in the following those $\boxplus$-sums, in which all summands $\Theta(P_{j,l})$ are indexed by distinct $(j,\Theta)\in\Omega$ with l depending on (j,Θ). For any such a $\boxplus$-sum, using the property that $\Theta(P_{j,l})\boxplus\Theta'(P_{j',l'})\sim\Theta(P_{j,l})$ for any $(j',\Theta')\prec(j,\Theta)$, we can remove one by one those $\boxplus$-summands $\Theta'(P_{j',l'})$ with (j',Θ') dominated by the index of another summand, without changing the equivalence class, until we reach a $\boxplus$-sum of $\Theta(P_{j,l})$ with $(j,\Theta)\in\Omega$ mutually unrelated by $\prec$, i.e. a reduced $\boxplus$-sum. So $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ consists of the reduced $\boxplus$-sums.

Note that for $(j,\Theta)\in\Omega$ and $l>0$,

$$\begin{aligned}\sigma_{(j,\Theta)}(\Theta(P_{j,l}))&=\sigma_{(j,\Theta)}\left(\boxplus^l\Theta\left(\left(\otimes^j I\right)\otimes\left(\otimes^{n-j}P_1\right)\right)\right)\\&=1\otimes\left(\boxplus^l\left(\otimes^{n-j}P_1\right)\right)\in C\left(\mathbb{T}^j\right)\otimes\left(\boxplus^l\mathcal{B}\left(\ell^2\left(\mathbb{Z}_{\geq}^{n-j}\right)\right)\right)\end{aligned}$$

and hence $\rho_{(j,\Theta)}\left(\Theta\left(P_{j,l}\right)\right) = l \in \mathbb{N}$ the operator rank of $\boxplus^l\left(\otimes^{n-j}P_1\right) \in \mathcal{B}\left(\oplus^l \ell^2\left(\mathbb{Z}_{\geq}^{n-j}\right)\right)$. But for $(j', \Theta') \neq (j, \Theta)$,

$$\rho_{(j,\Theta)}\left(\Theta'\left(P_{j',l'}\right)\right) := \begin{cases} \infty, & \text{if} \quad (j,\Theta) \prec (j',\Theta') \\ 0, & \text{if} \quad \text{otherwise} \end{cases}$$

because either $\sigma_{(j,\Theta)}\left(\Theta'\left(P_{j',l'}\right)\right) = 0$ when $\Theta\left(\{1,2,...,j\}\right) \setminus \Theta'\left(\{1,2,...,j'\}\right) \neq \emptyset$, or $\sigma_{(j,\Theta)}\left(\Theta'\left(P_{j',l'}\right)\right)$ is an infinite-dimensional projection when $\Theta'\left(\{1,2,...,j'\}\right) \supset \Theta\left(\{1,2,...,j\}\right)$ (but $\Theta\left(\{1,2,...,j\}\right) \neq \Theta'\left(\{1,2,...,j'\}\right)$ since $(j',\Theta') \neq (j,\Theta)$), i.e. when $(j,\Theta) \prec (j',\Theta')$.

For a reduced $\boxplus$-sum P of $\Theta'\left(P_{j',l'}\right)$ indexed by (j',Θ') in some subset $A \subset \Omega$, the (j,Θ)-component of $\rho(P)$ is

$$\sum_{(j',\Theta')\in A} \rho_{(j,\Theta)}\left(\Theta'\left(P_{j',l'}\right)\right) \begin{cases} = l \in \mathbb{N} & \text{if} \quad (j,\Theta) \in A \text{ with } \Theta\left(P_{j,l}\right) \text{ a summand of } P \\ \in \{0,\infty\} & \text{if} \quad \text{otherwise} \end{cases}$$

for any $(j,\Theta) \in \Omega$, since if $(j,\Theta) \in A$ then (j,Θ) is $\prec$-unrelated to any other $(j',\Theta') \in A$. So $\rho(P)$ completely determines the summands of a reduced $\boxplus$-sum P, namely, P is the $\boxplus$-sum of exactly those $\Theta\left(P_{j,l}\right)$ with l equal to the (j,Θ)-component of $\rho(P)$ that is a strictly positive integer. Since $\mathfrak{P}'\left(\mathcal{T}^{\otimes n}\right)$ consists of reduced $\boxplus$-sums, this also shows that the clearly well-defined monoid homomorphism ρ is injective.

Thus if $P \sim P'$ for two reduced $\boxplus$-sums P and P' and hence $\rho(P) = \rho(P')$, then the summands of P and P' are exactly the same, i.e. P and P' are the same reduced $\boxplus$-sum. So different reduced $\boxplus$-sums are mutually inequivalent projections.

□

Proposition 4. $\mathfrak{P}(\mathcal{T}) = \mathfrak{P}'(\mathcal{T})$. More concretely,

$$\mathfrak{P}(\mathcal{T}) \cong \{(0,l) : l \in \mathbb{Z}_{\geq}\} \cup \{(m,\infty) : m > 0\} \subset \overline{\mathbf{Z}_{\geq}}^2$$

where $\overline{\mathbf{Z}_{\geq}}^2$ is equipped with the canonical monoid structure.

Proof. It suffices to show that any element of $\mathfrak{P}_0(\mathcal{T}) \equiv \mathfrak{P}_0(\mathcal{T}^{\otimes 1})$ is of the form $P_{0,l}$ (realized as $(0,l) \in \overline{\mathbf{Z}_{\geq}}^2$) and any element of $\mathfrak{P}_m(\mathcal{T}) \equiv \mathfrak{P}_m(\mathcal{T}^{\otimes 1})$ for $m \in \mathbb{N}$ is of the form $P_{1,m}$ (realized as $(m,\infty) \in \overline{\mathbf{Z}_{\geq}}^2$).

The argument sketched below is similar to one used in [29].

Since any complex vector bundle over $\mathbb{T}$ is trivial, any idempotent over $C(\mathbb{T})$ is equivalent to the standard projection $1 \otimes P_m \in C(\mathbb{T}) \otimes M_\infty(\mathbb{C})$ for some $m \in \mathbb{Z}_{\geq}$. So for any idempotent $P \in M_\infty(\mathcal{T})$ over $\mathcal{T}$, there is some $U \in GL_\infty(C(\mathbb{T}))$ such that

$$U\sigma(P)U^{-1} = 1 \otimes P_m = \sigma(\boxplus^m I)$$

for some $m \in \mathbb{Z}_{\geq}$ where I is the identity of $\mathcal{K}^+ \subset \mathcal{T}$, and hence $VPV^{-1} - \boxplus^m I \in M_\infty(\mathcal{K})$ for any lift $V \in GL_\infty(\mathcal{T})$ (which exists) of $U \boxplus U^{-1} \in GL^0_\infty(C(\mathbb{T}))$ along σ. Replacing P by the equivalent VPV^{-1}, we may assume that $P \in (\boxplus^m I) + M_{k-1}(\mathcal{K}) \subset M_{k-1}(\mathcal{K}^+)$ for some large $k \geq m+1$. Now since $M_\infty(\mathbb{C})$ is dense in $\mathcal{K}$, there is an idempotent $Q \in (\boxplus^m I) + M_{k-1}(M_N(\mathbb{C}))$ sufficiently close to and hence equivalent to P for some large N. So replacing P by Q, we may assume that $K := P - \boxplus^m I \in M_{k-1}(M_N(\mathbb{C}))$.

Rearranging the entries of $P \equiv K + \boxplus^m I \in M_{k-1}(\mathcal{T}) \subset M_k(\mathcal{T})$ via conjugation by the unitary

$$U_{k,N} := \sum_{j=1}^{k-1} \left(e_{jj} \otimes (\mathcal{S}^*)^N + e_{kj} \otimes \left(\mathcal{S}^{(j-1)N} P_N\right)\right) + e_{kk} \otimes \mathcal{S}^{(k-1)N} \in M_k(\mathbb{C}) \otimes \mathcal{T} \equiv M_k(\mathcal{T})$$

we get

$$U_{k,N}PU_{k,N}^{-1} \equiv U_{k,N}\left(P \boxplus 0\right)U_{k,N}^{-1} = \left(\left(\boxplus^{m} I\right) \boxplus \left(\boxplus^{k-1-m} 0\right)\right) \boxplus R$$

for some $R \in M_{(k-1)N}\left(\mathbb{C}\right) \subset \mathcal{K} \subset \mathcal{T}$ which must be an idempotent. Since any idempotent in $\mathcal{K}$ is equivalent over $\mathcal{K}^{+}$ to a standard projection P_l, we get

$$P \sim \left(\left(\boxplus^{m} I\right) \boxplus \left(\boxplus^{k-1-m} 0\right)\right) \boxplus P_l$$

for some $l \in \mathbb{Z}_{\geq}$.

If $m = 0$, then clearly $P \sim P_l$. Since it is well known that P_l and $\boxplus^{l} P_1 \equiv P_{0,l}$ are equivalent over $\mathcal{K}^{+}$ and hence over $\mathcal{T} \supset \mathcal{K}^{+}$, we get $P \sim P_{0,l}$.

If $m \in \mathbb{N}$, then we can rearrange entries via conjugation by the unitary

$$U_l := e_{11} \otimes \mathcal{S}^{l} + e_{1k} \otimes P_l + \sum_{j=2}^{k-1} e_{jj} \otimes I + e_{kk} \otimes \left(\mathcal{S}^{*}\right)^{l} \in M_k\left(\mathbb{C}\right) \otimes \mathcal{T} \equiv M_k\left(\mathcal{T}\right)$$

to get

$$U_l\left(\left(\left(\boxplus^{m} I\right) \boxplus \left(\boxplus^{k-1-m} 0\right)\right) \boxplus P_l\right)U_l^{-1} = \left(\boxplus^{m} I\right) \boxplus \left(\boxplus^{k-m} 0\right) \equiv \boxplus^{m} I \equiv P_{1,m}.$$

□

Theorem 2. For $n > 1$ and $m > 0$, if $\mathfrak{P}_m\left(\mathcal{T}^{\otimes n-1}\right) = \mathfrak{P}'_m\left(\mathcal{T}^{\otimes n-1}\right) \equiv \left\{\boxplus^{m}\left(\otimes^{n-1} I\right)\right\}$ and $GL_m\left(\mathcal{T}^{\otimes n-1}\right)$ is connected, then $\mathfrak{P}_m\left(\mathcal{T}^{\otimes n}\right) = \mathfrak{P}'_m\left(\mathcal{T}^{\otimes n}\right)$.

Proof. In this proof, we use I and $\tilde{I}$ to denote respectively the identity elements of $\mathcal{T}^{\otimes n-1}$ and $\mathcal{T}^{\otimes n}$.

Let $P \in \mathfrak{P}_m\left(\mathcal{T}^{\otimes n}\right)$. The idempotent $\kappa_n\left(P\right)$ over $\mathcal{T}^{\otimes n-1} \otimes C\left(\mathbb{T}\right)$ satisfies that for any $z \in \mathbb{T}$,

$$\sigma_{n-1}\left(\kappa_n\left(P\right)\left(z\right)\right) = \sigma_n\left(P\right)\left(\cdot, z\right) \in M_{\infty}\left(C\left(\mathbb{T}^{n-1}\right)\right)$$

which is of rank m pointwise, and hence

$$\kappa_n(P)(z) \in \mathfrak{P}_m\left(\mathcal{T}^{\otimes n-1}\right) = \mathfrak{P}'_m\left(\mathcal{T}^{\otimes n-1}\right),$$

i.e. $\kappa_n(P)(z) \sim \boxplus^m I$ over $\mathcal{T}^{\otimes n-1}$. In particular, there is a continuous idempotent-valued path $\gamma : [0,1] \to M_k(\mathcal{T}^{\otimes n-1})$ for k sufficiently large going from the idempotent $\kappa_n(P)(1)$ to $(\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)$. Clearly we may assume that γ is locally constant at 1, say, $\gamma(t) = \boxplus^m I$ for $t \geq 1/2$. The concatenation of the path γ^{-1}, the loop $\kappa_n(P)$, and the path γ defines an idempotent-valued continuous loop $\Gamma : \mathbb{T} \to M_k(\mathcal{T}^{\otimes n-1})$ starting and ending at $\boxplus^m I$ with $\Gamma\left(e^{i\theta}\right) = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)$, say, for all $\theta \in [3\pi/2, 2\pi]$ (and $[0, \pi/2]$), and is homotopic to the loop $\kappa_n(P)$ via idempotents, i.e. there is a path of idempotents in $M_k(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ from $\kappa_n(P)$ to Γ. Consequently, there is a continuous path of invertibles $U_t \in GL_k(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ with $U_0 = I_k$ such that $U_1 \kappa_n(P) U_1^{-1} = \Gamma$, which can be lifted along κ_n to a continuous path of invertible $V_t \in GL_k(\mathcal{T}^{\otimes n})$ with $V_0 = I_k$ such that $\kappa_n\left(V_1 P V_1^{-1}\right) = \Gamma$.

Replacing P by the equivalent idempotent $V_1 P V_1^{-1}$, we may now assume directly that the idempotent $\kappa_n(P)$ over $\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is a continuous loop of idempotents in $M_k(\mathcal{T}^{\otimes n-1})$ such that $\kappa_n(P)\left(e^{i\theta}\right) = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)$ for all $\theta \in [3\pi/2, 2\pi]$. So there is a continuous path

$$\theta \in [0, 3\pi/2] \mapsto W_\theta \in GL_k\left(\mathcal{T}^{\otimes n-1}\right)$$

with $W_0 = I_k$ such that

$$W_\theta\left(\kappa_n(P)\left(e^{i\theta}\right)\right) W_\theta^{-1} = \kappa_n(P)(1) = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)$$

for all $\theta \in [0, 3\pi/2]$. In particular,

$$W_{3\pi/2}\left((\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)\right) = \left((\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)\right) W_{3\pi/2}$$

and hence $W_{3\pi/2} = W' \boxplus W''$ for some invertibles $W' \in GL_m\left(\mathcal{T}^{\otimes n-1}\right)$ and $W'' \in GL_{k-m}\left(\mathcal{T}^{\otimes n-1}\right)$.

By the connectedness assumption on $GL_m\left(\mathcal{T}^{\otimes n-1}\right)$, there is a continuous path $\alpha : [3\pi/2, 2\pi] \to GL_m\left(\mathcal{T}^{\otimes n-1}\right)$ with $\alpha(3\pi/2) = W'$ and $\alpha(2\pi) = I_m$. Since by Künneth formula, $K_1\left(\mathcal{T}^{\otimes n-1}\right) = 0$ and hence $GL_N\left(\mathcal{T}^{\otimes n-1}\right)$ is connected for N sufficiently large, we may suitably increase the value of k by adding diagonal $\boxplus$-summands 0 to idempotents and diagonal $\boxplus$-summands I to invertibles, so that $GL_{k-m}\left(\mathcal{T}^{\otimes n-1}\right)$ is also connected and hence there is a continuous path $\beta : [3\pi/2, 2\pi] \to GL_{k-m}\left(\mathcal{T}^{\otimes n-1}\right)$ with $\beta(3\pi/2) = W''$ and $\beta(2\pi) = I_{k-m}$. Now the function $\theta \mapsto W_\theta$ can be continuously extended to the whole interval $[0, 2\pi]$ by setting

$$W_\theta := \alpha(\theta) \boxplus \beta(\theta) \in GL_k\left(\mathcal{T}^{\otimes n-1}\right)$$

for $\theta \in [3\pi/2, 2\pi]$, giving rise to a well-defined continuous loop

$$W : e^{i\theta} \in \mathbb{T} \mapsto W_\theta \in GL_k\left(\mathcal{T}^{\otimes n-1}\right),$$

i.e. $W \in GL_k\left(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})\right)$, satisfying

$$W\left(\kappa_n(P)\right) W^{-1} = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right).$$

So the idempotent $\kappa_n(P)$ over $\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is equivalent to the idempotent $\boxplus^m I$.

Replacing P by the equivalent idempotent $\tilde{W}\left(P \boxplus \left(\boxplus^k 0\right)\right) \tilde{W}^{-1}$ for any fixed lifting $\tilde{W} \in GL_{2k}^0\left(\mathcal{T}^{\otimes n}\right)$ of $W \boxplus W^{-1} \in GL_{2k}^0\left(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})\right)$ along κ_n, we may now assume that

$$\kappa_n(P) = (\boxplus^m I) \boxplus \left(\boxplus^{2k-m} 0\right) = \kappa_n\left(\left(\boxplus^m \tilde{I}\right) \boxplus \left(\boxplus^{2k-m} 0\right)\right)$$

and proceed to show that $P \sim \boxplus^m \tilde{I}$.

Note that $P - \left(\left(\boxplus^m \tilde{I}\right) \boxplus \left(\boxplus^{2k-m} 0\right)\right) \in M_{2k}\left(\mathcal{T}^{\otimes n-1} \otimes \mathcal{K}\right)$. Since $M_\infty(\mathbb{C})$ is dense in $\mathcal{K}$, we may replace P by a suitable equivalent idempotent and assume that

$$K := P - \left(\left(\boxplus^m \tilde{I}\right) \boxplus \left(\boxplus^{2k-m} 0\right)\right) \in M_{2k}\left(\mathcal{T}^{\otimes n-1} \otimes M_N(\mathbb{C})\right) \subset M_{2k}\left(\mathcal{T}^{\otimes n}\right)$$

for some $N \in \mathbb{N}$.

Rearranging the entries of $P \equiv P \boxplus 0 \in M_{2k+1}\left(\mathcal{T}^{\otimes n-1} \otimes M_N(\mathbb{C})\right)$ via conjugation by the unitary

$$\begin{aligned} U_{k,N} &:= \sum_{j=1}^{2k} \left(e_{jj} \otimes (I \otimes \mathcal{S}^*)^N + e_{2k+1,j} \otimes \left(I \otimes \mathcal{S}^{(j-1)N} P_N\right)\right) + e_{2k+1,2k+1} \otimes \left(I \otimes \mathcal{S}^{2kN}\right) \\ &\in M_{2k+1}(\mathbb{C}) \otimes \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} \equiv M_{2k+1}\left(\mathcal{T}^{\otimes n}\right) \end{aligned}$$

we get

$$P \sim U_{k,N} P U_{k,N}^{-1} \equiv U_{k,N} (P \boxplus 0) U_{k,N}^{-1} = \left(\left(\boxplus^m \tilde{I}\right) \boxplus \left(\boxplus^{2k-m} 0\right)\right) \boxplus R$$

for some

$$R \in M_{2kN}\left(\mathcal{T}^{\otimes n-1}\right) \equiv \mathcal{T}^{\otimes n-1} \otimes M_{2kN}(\mathbb{C}) \subset \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} \equiv \mathcal{T}^{\otimes n}$$

which must be an idempotent over $\mathcal{T}^{\otimes n-1}$.

Since $K_0\left(\mathcal{T}^{\otimes n-1}\right) = \mathbb{Z}$ by Künneth formula, $R \boxplus \left(\boxplus^r I\right) \sim \left(\boxplus^{r+[R]} I\right)$ for a sufficiently large $r \in \mathbb{N}$ where $[R] \in \mathbb{Z}$ denotes the class of R in $K_0\left(\mathcal{T}^{\otimes n-1}\right)$. So there is an invertible $U \in GL_d\left(\mathcal{T}^{\otimes n-1}\right)$ for some large $d \geq \max\{2kN + r, r + [R]\}$ such that

$$U\left(R \boxplus \left(\boxplus^{d-2kN-r} 0\right) \boxplus \left(\boxplus^r I\right)\right) U^{-1} = \left(\boxplus^{d-r-[R]} 0\right) \boxplus \left(\boxplus^{r+[R]} I\right).$$

With $m>0$, we can rearrange entries via conjugation by the unitary

$$U_{d-r}:=e_{11}\otimes\left(I\otimes\mathcal{S}^{d-r}\right)+e_{1,2k+1}\otimes I\otimes P_{d-r}+\sum_{j=2}^{2k}e_{jj}\otimes\tilde{I}+e_{2k+1,2k+1}\otimes\left(I\otimes\mathcal{S}^{*}\right)^{d-r}$$

$$\in M_{2k+1}\left(\mathbb{C}\right)\otimes\mathcal{T}^{\otimes n-1}\otimes\mathcal{T}\equiv M_{2k+1}\left(\mathcal{T}^{\otimes n}\right)$$

to get

$$P\sim U_{d-r}\left(\left(\boxplus^{m}\tilde{I}\right)\boxplus\left(\boxplus^{2k-m}0\right)\boxplus R\right)U_{d-r}^{-1}=R'\boxplus\left(\boxplus^{m-1}\tilde{I}\right)\boxplus\left(\boxplus^{2k+1-m}0\right)$$

where

$$R'=\left(R\boxplus\left(\boxplus^{d-2kN-r}0\right)\right)+\left(\tilde{I}-I\otimes P_{d-r}\right)\in\tilde{I}+\left(\mathcal{T}^{\otimes n-1}\otimes M_{d-r}\left(\mathbb{C}\right)\right)$$

$$\subset\tilde{I}+\left(\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}\right)\subset\mathcal{T}^{\otimes n-1}\otimes\mathcal{T}=\mathcal{T}^{\otimes n}.$$

Note that R' can be interpreted as $R\boxplus\left(\boxplus^{d-2kN-r}0\right)\boxplus\left(\boxplus^{\infty}I\right)\in\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}^{+}\subset\mathcal{T}^{\otimes n}$, which when conjugated by the invertible $U\equiv U\boxplus\left(\boxplus^{\infty}I\right)\in\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}^{+}\subset\mathcal{T}^{\otimes n}$ becomes

$$\left(\boxplus^{d-r-[R]}0\right)\boxplus\left(\boxplus^{r+[R]}I\right)\boxplus\left(\boxplus^{\infty}I\right)=\tilde{I}-I\otimes P_{d-r-[R]}\in\tilde{I}+\left(\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}\right)\subset\mathcal{T}^{\otimes n}.$$

So we get

$$P\sim\left(\tilde{I}-I\otimes P_{d-r-[R]}\right)\boxplus\left(\boxplus^{m-1}\tilde{I}\right)\boxplus\left(\boxplus^{2k+1-m}0\right),$$

the latter of which when conjugated by $U_{d-r-[R]}^{-1}$ yields $\tilde{I}\boxplus\left(\boxplus^{m-1}\tilde{I}\right)\boxplus\left(\boxplus^{2k+1-m}0\right)$, where $U_{d-r-[R]}$ is defined as U_{d-r} by replacing $d-r$ by $d-r-[R]$. Thus we get $P\sim\left(\boxplus^{m}\tilde{I}\right)\boxplus\left(\boxplus^{2k+1-m}0\right)\equiv\boxplus^{m}\tilde{I}$.

□

Corollary 3. $\mathfrak{P}_m\left(\mathcal{T}^{\otimes n}\right)=\mathfrak{P}'_m\left(\mathcal{T}^{\otimes n}\right)\equiv\left\{\boxplus^{m}\tilde{I}\right\}$ for all $m\geq\left\lfloor\frac{n-1}{2}\right\rfloor+3$ and any $n\in\mathbb{N}$, where $\tilde{I}$ is the identity element of $\mathcal{T}^{\otimes n}$.

Proof. We prove by induction on $n \in \mathbb{N}$. For $n = 1$, we already know that $\mathfrak{P}'_m(\mathcal{T}^{\otimes n}) \equiv \mathfrak{P}_m(\mathcal{T}^{\otimes n})$ for all $m > 0$.

Now assume as the induction hypothesis that $\mathfrak{P}'_m(\mathcal{T}^{\otimes n}) = \mathfrak{P}_m(\mathcal{T}^{\otimes n})$ for all $m \geq \lfloor \frac{n-1}{2} \rfloor + 3$ for an $n \in \mathbb{N}$.

Since we know that $GL_m(\mathcal{T}^{\otimes n})$ is connected for all $m \geq \lfloor \frac{n}{2} \rfloor + 3$, the above theorem implies that $\mathfrak{P}'_m(\mathcal{T}^{\otimes n+1}) = \mathfrak{P}_m(\mathcal{T}^{\otimes n+1})$ for all $m \geq \lfloor \frac{n}{2} \rfloor + 3$.

□

It remains open the problem of classification of low-rank idempotents over $\mathcal{T}^{\otimes n}$. In particular, it is not clear whether there are idempotents of non-standard (equivalence) type.

6 Projective modules over $C\left(\mathbb{S}_H^{2n-1}\right)$

Most of the arguments and results in the above study of projective modules over $\mathcal{T}^{\otimes n}$ can be adapted to the case of the quantum spheres $C\left(\mathbb{S}_H^{2n-1}\right)$.

Let $\partial_n : \mathcal{T}^{\otimes n} \to C\left(\mathbb{S}_H^{2n-1}\right)$ be the canonical quotient map by restricting the groupoid $\mathfrak{T}_n$ to the closed invariant set $\overline{\mathbb{Z}_\geq}^n \backslash \mathbb{Z}_\geq^n$ in its unit space.

First we note that there is a short exact sequence of C*-algebras

$$0 \to C\left(\mathbb{S}_H^{2n-3}\right) \otimes \mathcal{K}\left(\ell^2\left(\mathbb{Z}_\geq\right)\right) \to C\left(\mathbb{S}_H^{2n-1}\right) \xrightarrow{\lambda_n} C^*\left(\mathfrak{T}_n|_{\overline{\mathbb{Z}_\geq}^{n-1} \times \{\infty\}}\right) \cong \mathcal{T}^{\otimes n-1} \otimes C\left(\mathbb{T}\right) \to 0$$

for all $n > 1$. Indeed, since $\left(\overline{\mathbb{Z}_\geq}^{n-1} \backslash \mathbb{Z}_\geq^{n-1}\right) \times \mathbb{Z}_\geq$ is an open invariant subset of the unit space $\overline{\mathbb{Z}_\geq}^n \backslash \mathbb{Z}_\geq^n$ of the groupoid $\mathfrak{G}_n \equiv \left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)|_{\overline{\mathbb{Z}_\geq}^n \backslash \mathbb{Z}_\geq^n}$ with the invariant complement

$$\left(\overline{\mathbb{Z}_\geq}^n \backslash \mathbb{Z}_\geq^n\right) \setminus \left(\left(\overline{\mathbb{Z}_\geq}^{n-1} \backslash \mathbb{Z}_\geq^{n-1}\right) \times \mathbb{Z}_\geq\right) = \overline{\mathbb{Z}_\geq}^{n-1} \times \{\infty\},$$

the groupoid C*-algebra

$$C^*\left(\mathfrak{G}_n|_{\left(\overline{\mathbb{Z}_\geq}^{n-1}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\mathbb{Z}_\geq}\right)=C^*\left(\left(\mathbb{Z}^{n-1}\ltimes\overline{\mathbb{Z}}^{n-1}\right)\Big|_{\overline{\mathbb{Z}_\geq}^{n-1}\setminus\mathbb{Z}_\geq^{n-1}}\times\left(\mathbb{Z}\ltimes\mathbb{Z}\right)|_{\mathbb{Z}_\geq}\right)$$

$$\cong C^*\left(\left(\mathbb{Z}^{n-1}\ltimes\overline{\mathbb{Z}}^{n-1}\right)\Big|_{\overline{\mathbb{Z}_\geq}^{n-1}\setminus\mathbb{Z}_\geq^{n-1}}\right)\otimes C^*\left(\left(\mathbb{Z}\ltimes\mathbb{Z}\right)|_{\mathbb{Z}_\geq}\right)=C\left(\mathbb{S}_H^{2n-3}\right)\otimes\mathcal{K}\left(\ell^2\left(\mathbb{Z}_\geq\right)\right)$$

is a closed ideal of $C^*\left(\mathfrak{G}_n\right)=C\left(\mathbb{S}_H^{2n-1}\right)$ with quotient

$$C^*\left(\mathfrak{G}_n\right)/C^*\left(\mathfrak{G}_n|_{\left(\overline{\mathbb{Z}_\geq}^{n-1}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\mathbb{Z}_\geq}\right)\cong C^*\left(\mathfrak{G}_n|_{\overline{\mathbb{Z}_\geq}^{n-1}\times\{\infty\}}\right)$$

$$=C^*\left(\left(\mathbb{Z}^{n-1}\ltimes\overline{\mathbb{Z}}^{n-1}\right)\Big|_{\overline{\mathbb{Z}_\geq}^{n-1}}\times\mathbb{Z}\right)\cong C^*\left(\left(\mathbb{Z}^{n-1}\ltimes\overline{\mathbb{Z}}^{n-1}\right)\Big|_{\overline{\mathbb{Z}_\geq}^{n-1}}\right)\otimes C\left(\mathbb{T}\right)=\mathcal{T}^{\otimes n-1}\otimes C\left(\mathbb{T}\right).$$

So we get the above short exact sequence with λ_n being the canonical map from $C^*\left(\mathfrak{G}_n\right)$ to its quotient $\mathcal{T}^{\otimes n-1}\otimes C\left(\mathbb{T}\right)$ resulting from restricting the groupoid $\mathfrak{G}_n$ to the closed invariant set $\overline{\mathbb{Z}_\geq}^{n-1}\times\{\infty\}$.

Clearly $\kappa_n=\lambda_n\circ\partial_n$. Furthermore all the quotient maps $\sigma_{(j,\Theta)}$ on $\mathcal{T}^{\otimes n}$ with $j>0$ factors through ∂_n and induces a quotient map

$$\tau_{(j,\Theta)}:C\left(\mathbb{S}_H^{2n-1}\right)\rightarrow C\left(\mathbb{T}^j\right)\otimes\mathcal{B}\left(\ell^2\left(\mathbb{Z}_\geq^{n-j}\right)\right)$$

such that $\sigma_{(j,\Theta)}=\tau_{(j,\Theta)}\circ\partial_n$.

Note that the quotient maps λ_n for $n\in\mathbb{N}$ satisfy the commuting diagram

$$\begin{array}{ccccc}M_k\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) & \overset{\lambda_n}{\rightarrow} & M_k\left(\mathcal{T}^{\otimes n-1}\otimes C\left(\mathbb{T}\right)\right) & \equiv & M_k\left(\mathcal{T}^{\otimes n-1}\right)\otimes C\left(\mathbb{T}\right)\\ \downarrow_{\tau_n} & \circlearrowright & \downarrow_{\sigma_{n-1}\otimes\mathrm{id}} & & \downarrow_{\sigma_{n-1}\otimes\mathrm{id}}\\ M_k\left(C\left(\mathbb{T}^n\right)\right) & \overset{\equiv}{\rightarrow} & M_k\left(C\left(\mathbb{T}^{n-1}\right)\otimes C\left(\mathbb{T}\right)\right) & \equiv & M_k\left(C\left(\mathbb{T}^{n-1}\right)\right)\otimes C\left(\mathbb{T}\right).\end{array}$$

We define the rank of (the equivalence class of) an idempotent $Q\in M_\infty\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ over $C\left(\mathbb{S}_H^{2n-1}\right)$ as the rank of the matrix value $\tau_n\left(Q\right)\left(z\right)\in M_\infty\left(\mathbb{C}\right)$ at any $z\in\mathbb{T}^n$ (independent of z since $\mathbb{T}^n$ is connected). Then the set of equivalence classes of idempotents

$Q \in M_\infty\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ equipped with the binary operation $\boxplus$ becomes an abelian graded monoid

$$\mathfrak{P}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) = \sqcup_{m=0}^{\infty} \mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$$

where $\mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ is the set of all (equivalence classes of) idempotents over $C\left(\mathbb{S}_H^{2n-1}\right)$ of rank m, with clearly

$$\mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \boxplus \mathfrak{P}_l\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \subset \mathfrak{P}_{m+l}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$$

for $m, l \geq 0$.

Since $\sigma_n = \tau_n \circ \partial_n$, the rank of an idempotent P over $C\left(\mathcal{T}^{\otimes n}\right)$ equals the rank of the idempotent $\partial_n P$ over $C\left(\mathbb{S}_H^{2n-1}\right)$. We now define

$$\mathfrak{P}'_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) := \partial_n\left(\mathfrak{P}'_m\left(\mathcal{T}^{\otimes n}\right)\right) \subset \mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right),$$

and the projections

$$Q_{j,\Theta,l} := \partial_n\left(\Theta\left(P_{j,l}\right)\right)$$

over $C\left(\mathbb{S}_H^{2n-1}\right)$. Note that $\mathfrak{P}'_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) = \left\{\boxplus^m \tilde{I}\right\}$ for $m > 0$, where $\tilde{I}$ denotes the identity element of $C\left(\mathbb{S}_H^{2n-1}\right)$.

Also note that $Q_{0,\mathrm{id},l} = 0$ for all l, where $\mathrm{id} \equiv \mathrm{id}_{\{1,2,\ldots,n\}}$ is the only $(0,n)$-shuffle. The monoid homomorphism

$$\rho_0 : P \in \mathfrak{P}'\left(\mathcal{T}^{\otimes n}\right) \mapsto \prod_{(j,\Theta)\in\Omega_0} \rho_{(j,\Theta)}(P) \in \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}_{\geq}},$$

with

$$\Omega_0 := \Omega \backslash \{(0,\mathrm{id})\} \equiv \{(j,\Theta) : \ 0 < j \leq n \text{ and } \Theta \text{ is a } (j, n-j)\text{-shuffle}\},$$

"truncated" from ρ induces a well-defined monoid homomorphism

$$\rho_\partial : \mathfrak{P}'\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \to \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}_\geq},$$

in the sense that $\rho = \rho_\partial \circ \partial_n$. Indeed for $(j,\Theta) \in \Omega_0$, i.e. with $j > 0$, the quotient map

$$\sigma_{(j,\Theta)} : \mathcal{T}^{\otimes n} \equiv C^*\left(\mathfrak{T}_n\right) \to C^*\left(\mathfrak{T}_n|_{X_\Theta}\right)$$

factors through ∂_n since the unit space $\overline{\mathbb{Z}_\geq^n}\backslash\mathbb{Z}_\geq^n$ of $\mathfrak{G}_n$ contains X_Θ, and hence the map $\rho_{(j,\Theta)}$ factors through ∂_n.

We call a $\boxplus$-sum of $Q_{j,\Theta,l}$ indexed by $\prec$-unrelated $(j,\Theta) \in \Omega_0$ (i.e. $1 \leq j \leq n$) and $l \equiv l_{(j,\Theta)} > 0$ depending on (j,Θ) to be a reduced $\boxplus$-sum of standard projections over $C\left(\mathbb{S}_H^{2n-1}\right)$. (The degenerate empty $\boxplus$-sum 0 is taken as a reduced $\boxplus$-sum.) Two such reduced $\boxplus$-sums are called different when they have different sets of (mutually $\prec$-unrelated) indices $(j,\Theta) \in \Omega_0$ or have different weight functions l of (j,Θ). Each $Q_{j,\Theta,l}$ with $j, l > 0$ is a reduced $\boxplus$-sum of standard projections over $C\left(\mathbb{S}_H^{2n-1}\right)$.

Proposition 5. Different reduced $\boxplus$-sums of standard projections over $C\left(\mathbb{S}_H^{2n-1}\right)$ are mutually inequivalent projections over $C\left(\mathbb{S}_H^{2n-1}\right)$, and they form a graded submonoid

$$\mathfrak{P}'\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) = \sqcup_{m=0}^{\infty} \mathfrak{P}'_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$$

of the monoid $\mathfrak{P}\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$, with its monoid structure explicitly determined by $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ for $(j',\Theta') \prec (j,\Theta)$ with $j, j', l, l' > 0$. Furthermore the monoid homomorphism

$$\rho_\partial : \mathfrak{P}'\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) \to \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}_\geq}$$

is injective.

Proof. The submonoid $\mathfrak{P}'\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) = \partial_n\left(\mathfrak{P}'\left(C\left(\mathcal{T}^{\otimes n}\right)\right)\right)$ consists of reduced $\boxplus$-sums of $Q_{j,\Theta,l} = \partial_n\left(\Theta\left(P_{j,l}\right)\right)$ with $j>0$, since $Q_{0,\mathrm{id},l}=0$.

Let $\mathfrak{M}$ be the subset of $\mathfrak{P}'\left(C\left(\mathcal{T}^{\otimes n}\right)\right)$ consisting of all reduced $\boxplus$-sums P of $\Theta\left(P_{j,l}\right)$ with $j>0$. Then $\partial_n|_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{P}'\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ is still surjective, and $\rho_0|_{\mathfrak{M}}$ still factors through ρ_∂, i.e. $\rho_0|_{\mathfrak{M}} = \rho_\partial \circ \partial_n|_{\mathfrak{M}}$. These imply that ρ_∂ is injective if $\rho_0|_{\mathfrak{M}}$ is injective.

For any reduced $\boxplus$-sum $P \in \mathfrak{M}$ of $\Theta\left(P_{j,l}\right)$ with $j>0$, the (j,Θ)-component of $\rho\left(P\right)$ is the same as that of $\rho_0\left(P\right)$ for all $(j,\Theta)\in\Omega_0$, while the only other component, namely, the $(0,\mathrm{id})$-component of $\rho\left(P\right)$ is ∞ since $\rho_{(0,\mathrm{id})}\left(\Theta\left(P_{j,l}\right)\right)=\infty$ for any $j>0$. Thus we get $\rho\left(P\right) = \left(\infty, \rho_0\left(P\right)\right)$ for all $P\in\mathfrak{M}$. Hence the injectivity of $\rho|_{\mathfrak{M}}$ implies the injectivity of $\rho_0|_{\mathfrak{M}}$ on $\mathfrak{M}$, and hence the injectivity of ρ_∂.

Since two different reduced $\boxplus$-sums Q, Q' over $C\left(\mathbb{S}_H^{2n-1}\right)$ are of the form $\partial_n\left(P\right), \partial_n\left(P'\right)$ respectively for two different reduced $\boxplus$-sums $P, P' \in \mathfrak{M}$ over $C\left(\mathcal{T}^{\otimes n}\right)$ which are inequivalent over $C\left(\mathcal{T}^{\otimes n}\right)$ and hence $\rho_0\left(P\right) \neq \rho_0\left(P'\right)$, we get $\rho_\partial\left(Q\right) \neq \rho_\partial\left(Q'\right)$ showing that Q, Q' are different equivalence classes in $\mathfrak{P}'\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$.

The property that $\Theta\left(P_{j,l}\right) \boxplus \Theta'\left(P_{j',l'}\right) \sim \Theta\left(P_{j,l}\right)$ over $\mathcal{T}^{\otimes n}$ for $(j',\Theta') \prec (j,\Theta)$ is clearly preserved under the quotient map ∂_n, i.e. $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ over $C\left(\mathbb{S}_H^{2n-1}\right)$.

□

Theorem 3. For $n>1$ and $m\in\mathbb{N}$, if $\mathfrak{P}_m\left(\mathcal{T}^{\otimes n-1}\right) = \mathfrak{P}'_m\left(\mathcal{T}^{\otimes n-1}\right)$ and $GL_m\left(\mathcal{T}^{\otimes n-1}\right)$ is connected, then $\mathfrak{P}'_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right) = \mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$.

Proof. Many arguments used to prove a similar theorem for $\mathcal{T}^{\otimes n}$ instead of $C\left(\mathbb{S}_H^{2n-1}\right)$

can be used again here with minor modifications. In this proof, I and $\tilde{I}$ denote respectively the identity element of $\mathcal{T}^{\otimes n-1}$ and $C\left(\mathbb{S}_H^{2n-1}\right)$.

Let $P \in \mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$. The idempotent $\lambda_n(P)$ over $\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})$ satisfies that for any $z \in \mathbb{T}$,

$$\sigma_{n-1}\left(\lambda_n(P)(z)\right) = \tau_n(P)(\cdot, z) \in M_\infty\left(C\left(\mathbb{T}^{n-1}\right)\right)$$

which is of rank m pointwise, and hence

$$\lambda_n(P)(z) \in \mathfrak{P}_m\left(\mathcal{T}^{\otimes n-1}\right) = \mathfrak{P}'_m\left(\mathcal{T}^{\otimes n-1}\right),$$

i.e. $\lambda_n(P)(z) \sim \boxplus^m I$ over $\mathcal{T}^{\otimes n-1}$. As before, for some large k, there is an idempotent-valued continuous loop $\Gamma : \mathbb{T} \to M_k(\mathcal{T}^{\otimes n-1})$ starting and ending at $\boxplus^m I$ with $\Gamma\left(e^{i\theta}\right) = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)$, say, for all $\theta \in [3\pi/2, 2\pi]$, and homotopic to the loop $\lambda_n(P)$ via idempotents. Consequently, there is a continuous path of invertibles $U_t \in GL_k(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ with $U_0 = I_k$ such that $U_1 \lambda_n(P) U_1^{-1} = \Gamma$, which can be lifted along λ_n to a continuous path of invertible $V_t \in GL_k\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ with $V_0 = I_k$ such that $\lambda_n\left(V_1 P V_1^{-1}\right) = \Gamma$.

Replacing P by the equivalent idempotent $V_1 P V_1^{-1}$, we may now assume directly that the idempotent $\lambda_n(P)$ over $\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is a continuous loop of idempotents in $M_k(\mathcal{T}^{\otimes n-1})$ such that $\lambda_n(P)\left(e^{i\theta}\right) = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right)$ for all $\theta \in [3\pi/2, 2\pi]$. As before, by the connectedness assumption on $GL_m(\mathcal{T}^{\otimes n-1})$, after suitably increasing the size k, we can find a well-defined continuous loop

$$W : e^{i\theta} \in \mathbb{T} \mapsto W_\theta \in GL_k\left(\mathcal{T}^{\otimes n-1}\right),$$

i.e. $W \in GL_k(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T}))$, satisfying

$$W\left(\lambda_n(P)\right) W^{-1} = (\boxplus^m I) \boxplus \left(\boxplus^{k-m} 0\right).$$

So the idempotent $\lambda_n(P)$ over $\mathcal{T}^{\otimes n-1}\otimes C(\mathbb{T})$ is equivalent to the idempotent $\boxplus^m I$.

Replacing P by the equivalent idempotent $\tilde{W}\left(P\boxplus\left(\boxplus^k 0\right)\right)\tilde{W}^{-1}$ for any fixed lifting $\tilde{W}\in GL^0_{2k}\left(C\left(\mathbb{S}^{2n-1}_H\right)\right)$ of $W\boxplus W^{-1}\in GL^0_{2k}\left(\mathcal{T}^{\otimes n-1}\otimes C(\mathbb{T})\right)$ along λ_n, we may now assume that

$$\lambda_n(P)=\left(\boxplus^m I\right)\boxplus\left(\boxplus^{2k-m}0\right)=\lambda_n\left(\left(\boxplus^m\tilde{I}\right)\boxplus\left(\boxplus^{2k-m}0\right)\right)$$

and proceed to show that $P\sim\boxplus^m\tilde{I}$ over $C\left(\mathbb{S}^{2n-1}_H\right)$, where we use $\tilde{I}$ to denote the identity element in $C\left(\mathbb{S}^{2n-1}_H\right)$ so as to distinguish it from the identity element I of $\mathcal{T}^{\otimes n-1}$.

With $P-\left(\left(\boxplus^m\tilde{I}\right)\boxplus\left(\boxplus^{2k-m}0\right)\right)\in M_{2k}\left(C\left(\mathbb{S}^{2n-3}_H\right)\otimes\mathcal{K}\right)$ and $M_\infty(\mathbb{C})$ dense in $\mathcal{K}$, we may replace P by a suitable equivalent idempotent and assume that

$$P=K+\left(\left(\boxplus^m\tilde{I}\right)\boxplus\left(\boxplus^{2k-m}0\right)\right)\in M_{2k}\left(\left(C\left(\mathbb{S}^{2n-3}_H\right)\otimes\mathcal{K}\right)^+\right)\subset M_{2k}\left(C\left(\mathbb{S}^{2n-1}_H\right)\right)$$

for some $K\in M_{2k}\left(C\left(\mathbb{S}^{2n-3}_H\right)\otimes M_N(\mathbb{C})\right)$ and some $N\in\mathbb{N}$.

As before, by rearranging entries via conjugation, we get

$$P\sim\partial_n\left(U_{k,N}\right)P\partial_n\left(U^{-1}_{k,N}\right)\equiv\partial_n\left(U_{k,N}\right)\left(P\boxplus 0\right)\partial_n\left(U^{-1}_{k,N}\right)=\left(\left(\boxplus^m\tilde{I}\right)\boxplus\left(\boxplus^{2k-m}0\right)\right)\boxplus R$$

for some

$$R\in M_{2kN}\left(C\left(\mathbb{S}^{2n-3}_H\right)\right)\equiv C\left(\mathbb{S}^{2n-3}_H\right)\otimes M_{2kN}(\mathbb{C})\subset\left(C\left(\mathbb{S}^{2n-3}_H\right)\otimes\mathcal{K}\right)^+\subset C\left(\mathbb{S}^{2n-1}_H\right)$$

which must be an idempotent over $C\left(\mathbb{S}^{2n-3}_H\right)$. More precisely, we can lift P to

$$\hat{P}=\hat{K}+\left(\left(\boxplus^m I_{\mathcal{T}^{\otimes n}}\right)\boxplus\left(\boxplus^{2k-m}0\right)\right)\in M_{2k}\left(\left(\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}\right)^+\right)\subset M_{2k}\left(\mathcal{T}^{\otimes n}\right)$$

for some $\hat{K}\in M_{2k}\left(\mathcal{T}^{\otimes n-1}\otimes M_N(\mathbb{C})\right)$ and conjugate it by the unitary $U_{k,N}$ over $\mathcal{T}^{\otimes n}$ to get the form $\left(\left(\boxplus^m I_{\mathcal{T}^{\otimes n}}\right)\boxplus\left(\boxplus^{2k-m}0\right)\right)\boxplus\hat{R}$ with $\hat{R}\in M_{2kN}\left(\mathcal{T}^{\otimes n-1}\right)$ as we did for the case of

$\mathcal{T}^{\otimes n}$. Then the above R is $\partial_n\left(\hat{R}\right)$. Note that even though $\hat{P}$ and $\hat{R}$ are not necessarily idempotents, R is since it is the idempotent P conjugated by the unitary $\partial_n\left(U_{k,N}\right)$ over $C\left(\mathbb{S}_H^{2n-1}\right)$.

Since $K_0\left(C\left(\mathbb{S}_H^{2n-3}\right)\right) = \mathbb{Z}$ [12], $R \boxplus \left(\boxplus^r \hat{I}\right) \sim \left(\boxplus^{r+[R]}\hat{I}\right)$ for a sufficiently large $r \in \mathbb{N}$ where $[R] \in \mathbb{Z}$ denotes the class of R in $K_0\left(C\left(\mathbb{S}_H^{2n-3}\right)\right)$ and $\hat{I}$ is the identity element of $C\left(\mathbb{S}_H^{2n-3}\right)$. So there is an invertible $U \in GL_d\left(C\left(\mathbb{S}_H^{2n-3}\right)\right)$ for some large $d \geq \max\{2kN+r, r+[R]\}$ such that

$$U\left(R \boxplus \left(\boxplus^{d-2kN-r}0\right) \boxplus \left(\boxplus^r \hat{I}\right)\right)U^{-1} = \left(\boxplus^{d-r-[R]}0\right) \boxplus \left(\boxplus^{r+[R]}\hat{I}\right).$$

As before, with $m > 0$, by rearranging entries via conjugation, we can get

$$P \sim R' \boxplus \left(\boxplus^{m-1}\tilde{I}\right) \boxplus \left(\boxplus^{2k+1-m}0\right)$$

where the idempotent

$$R' = \left(R \boxplus \left(\boxplus^{d-2kN-r}0\right)\right) + \left(\tilde{I} - \hat{I} \otimes P_{d-r}\right) \in \left(C\left(\mathbb{S}_H^{2n-3}\right) \otimes \mathcal{K}\right)^+ \subset C\left(\mathbb{S}_H^{2n-1}\right)$$

when conjugated by the invertible $U \equiv U \boxplus \left(\tilde{I} - \hat{I} \otimes P_d\right) \in \left(C\left(\mathbb{S}_H^{2n-3}\right) \otimes \mathcal{K}\right)^+$ becomes

$$\left(\boxplus^{d-r-[R]}0\right) \boxplus \left(\tilde{I} - \hat{I} \otimes P_{d-r-[R]}\right) \in \left(C\left(\mathbb{S}_H^{2n-3}\right) \otimes \mathcal{K}\right)^+ \subset C\left(\mathbb{S}_H^{2n-1}\right).$$

So we get

$$P \sim \left(\left(\boxplus^{d-r-[R]}0\right) \boxplus \left(\tilde{I} - \hat{I} \otimes P_{d-r-[R]}\right)\right) \boxplus \left(\boxplus^{m-1}\tilde{I}\right) \boxplus \left(\boxplus^{2k+1-m}0\right),$$

the latter of which as before is equivalent to $\tilde{I} \boxplus \left(\boxplus^{m-1}\tilde{I}\right) \boxplus \left(\boxplus^{2k+1-m}0\right)$ by a further conjugation by $U_{d-r-[R]}^{-1}$. Thus $P \sim \left(\boxplus^m \tilde{I}\right) \boxplus \left(\boxplus^{2k+1-m}0\right) \equiv \boxplus^m \tilde{I}$.

□

Corollary 4. $\mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)=\mathfrak{P}'_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)\equiv\left\{\boxplus^m\tilde{I}\right\}$ for all $m\geq\left\lfloor\frac{n-1}{2}\right\rfloor+3$ and any $n\in\mathbb{N}$, where $\tilde{I}$ is the identity element of $C\left(\mathbb{S}_H^{2n-1}\right)$.

Proof. The case of $n=1$ is well known. For $n>1$, since $\mathfrak{P}'_m\left(\mathcal{T}^{\otimes n-1}\right)=\mathfrak{P}_m\left(\mathcal{T}^{\otimes n-1}\right)$ for all $m\geq\left\lfloor\frac{n-2}{2}\right\rfloor+3$ and $GL_m\left(\mathcal{T}^{\otimes n-1}\right)$ is connected for all $m\geq\left\lfloor\frac{n-1}{2}\right\rfloor+3$, the above theorem implies that $\mathfrak{P}'_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)=\mathfrak{P}_m\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)$ for all $m\geq\left\lfloor\frac{n-1}{2}\right\rfloor+3$.

□

It is not clear whether there are (low-rank) idempotents over $C\left(\mathbb{S}_H^{2n-1}\right)$ of non-standard (equivalence) type and whether the cancellation law holds for them.

7 Projective modules over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$

In this section we study the problem of classification of finitely generated projective modules over the multipullback quantum complex projective space $\mathbb{P}^{n-1}\left(\mathcal{T}\right)$ that was introduced and studied by Hajac, Kaygun, Zieliński in [9].

In [12], $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)=\mathbb{Z}^n$ and $K_1\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)=0$ are computed, and $\mathbb{P}^{n-1}\left(\mathcal{T}\right)$ is shown to be a quantum quotient space of $\mathbb{S}_H^{2n-1}$. More precisely, the C*-algebra $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ is isomorphic to the invariant C*-subalgebra $\left(C\left(\mathbb{S}_H^{2n-1}\right)\right)^{U(1)}$ of $C\left(\mathbb{S}_H^{2n-1}\right)$ under the canonical diagonal $U\left(1\right)$-action on $C\left(\mathbb{S}_H^{2n-1}\right)\cong\mathcal{T}^{\otimes n}/\mathcal{K}^{\otimes n}$, which in the groupoid context can be implemented by the multiplication operator

$$U_\zeta: f\in C_c\left(\mathfrak{G}_n\right)\mapsto h_\zeta f\in C_c\left(\mathfrak{G}_n\right)$$

for $\zeta \in U(1) \equiv \mathbb{T}$ where

$$h_\zeta : (m,p) \in \mathfrak{G}_n \subset \mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n \mapsto \zeta^{\Sigma m} \in \mathbb{T} \text{ with } \Sigma m := \sum_{i=1}^n m_i$$

is a groupoid character. Then $C(\mathbb{P}^{n-1}(\mathcal{T}))$ is realized as the groupoid C*-algebra $C^*((\mathfrak{G}_n)_0)$ of the subgroupoid $(\mathfrak{G}_n)_0$ of $\mathfrak{G}_n$, where

$$(\mathfrak{G}_n)_k := \{(m,p) \in \mathfrak{G}_n : \Sigma m = k\}$$

for $k \in \mathbb{Z}$. Furthermore, $C^*(\mathfrak{G}_n)$ becomes a (completion of the) graded algebra $\oplus_{k\in\mathbb{Z}}\overline{C_c((\mathfrak{G}_n)_k)}$ with the component $\overline{C_c((\mathfrak{G}_n)_k)}$ being the quantum line bundle $C\left(\mathbb{S}_H^{2n-1}\right)_k$ [12] of degree k over the quantum space $\mathbb{P}^{n-1}(\mathcal{T})$.

It is easy to see that the standard projections $Q_{j,\Theta,l} \equiv \partial_n(\Theta(P_{j,l}))$ over $C\left(\mathbb{S}_H^{2n-1}\right)$ with $j,l>0$ found in the previous section lie in $M_\infty(C^*((\mathfrak{G}_n)_0))$ since $P_{j,l} = \boxplus^l((\otimes^j I) \otimes (\otimes^{n-j} P_1))$ is in $C^*((\mathfrak{T}_n)_0)$, and hence are also projections over $C^*((\mathfrak{G}_n)_0) \equiv C(\mathbb{P}^{n-1}(\mathcal{T}))$. Furthermore with $C(\mathbb{P}^{n-1}(\mathcal{T})) \subset C\left(\mathbb{S}_H^{2n-1}\right)$, inequivalent $\boxplus$-sums of standard projections over $C\left(\mathbb{S}_H^{2n-1}\right)$ must be inequivalent over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ as well.

Proposition 6. Different reduced $\boxplus$-sums of standard projections $Q_{j,\Theta,l}$ over $C\left(\mathbb{S}_H^{2n-1}\right)$ with $j,l>0$ when viewed as projections over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ are mutually inequivalent over $C(\mathbb{P}^{n-1}(\mathcal{T}))$, and they form a graded submonoid

$$\mathfrak{P}'\left(C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\right) = \sqcup_{m=0}^{\infty}\mathfrak{P}'_m\left(C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\right)$$

of the monoid $\mathfrak{P}(C(\mathbb{P}^{n-1}(\mathcal{T})))$. Furthermore the monoid homomorphism

$$\mathfrak{P}'\left(C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\right) \to \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}_\geq}$$

inherited from ρ_∂ is injective.

However, for $(j',\Theta') \prec (j,\Theta)$ with $j,j',l,l' > 0$, it is no longer true in general that $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$, even though $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ over $C\left(\mathbb{S}_H^{2n-1}\right)$ since the invertible matrix over $C\left(\mathbb{S}_H^{2n-1}\right)$ intertwining $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'}$ and $Q_{j,\Theta,l}$ may not be replaced by one over the subalgebra $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ of $C\left(\mathbb{S}_H^{2n-1}\right)$.

In the following, we show that the standard projections $Q_{j,\mathrm{id},1}$ with $j > 0$ provide a set of representatives of K_0-classes that freely generate the abelian K_0-group of $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$.

The subgroupoid $\mathfrak{H}_j := \mathfrak{G}_j \times \left(\mathbb{Z}^{n-j} \ltimes \mathbb{Z}_{\geq}^{n-j}\right)$ of $\mathfrak{G}_n$ for $1 \leq j \leq n$ is the groupoid $\mathfrak{G}_n$ restricted to the open invariant subset $\left(\overline{\mathbb{Z}_{\geq}^{j}} \times \mathbb{Z}_{\geq}^{n-j}\right) \backslash \mathbb{Z}_{\geq}^{n}$ and inherits the grading of $\mathfrak{G}_n$. The grade-0 part $\left(\mathfrak{H}_j\right)_0$ of $\mathfrak{H}_j$ is the groupoid $\left(\mathfrak{G}_n\right)_0$ restricted to $\left(\overline{\mathbb{Z}_{\geq}^{j}} \times \mathbb{Z}_{\geq}^{n-j}\right) \backslash \mathbb{Z}_{\geq}^{n}$, and from the increasing chain of $\left(\mathfrak{H}_j\right)_0$, we get an increasing composition sequence of closed ideals of $C^*\left(\left(\mathfrak{G}_n\right)_0\right)$ as

$$0 =: C^*\left(\left(\mathfrak{H}_0\right)_0\right) \lhd C^*\left(\left(\mathfrak{H}_1\right)_0\right) \lhd \cdots \lhd C^*\left(\left(\mathfrak{H}_{n-1}\right)_0\right) \lhd C^*\left(\left(\mathfrak{H}_n\right)_0\right) = C^*\left(\left(\mathfrak{G}_n\right)_0\right)$$

such that with $\left(\overline{\mathbb{Z}_{\geq}^{j}} \times \mathbb{Z}_{\geq}^{n-j}\right) \backslash \left(\overline{\mathbb{Z}_{\geq}^{j-1}} \times \mathbb{Z}_{\geq}^{n-j+1}\right) = \overline{\mathbb{Z}_{\geq}^{j-1}} \times \{\infty\} \times \mathbb{Z}_{\geq}^{n-j}$,

$$C^*\left(\left(\mathfrak{H}_j\right)_0\right) / C^*\left(\left(\mathfrak{H}_{j-1}\right)_0\right) \cong C^*\left(\left(\mathfrak{G}_n|_{\overline{\mathbb{Z}_{\geq}^{j-1}} \times \{\infty\} \times \mathbb{Z}_{\geq}^{n-j}}\right)_0\right) \cong C^*\left(\mathfrak{T}_{n-1}|_{\overline{\mathbb{Z}_{\geq}^{j-1}} \times \mathbb{Z}_{\geq}^{n-j}}\right) \cong \mathcal{T}^{\otimes j-1} \otimes \mathcal{K}\left(\mathbb{Z}_{\geq}^{n-j}\right)$$

because the groupoid $\left(\mathfrak{G}_n|_{\overline{\mathbb{Z}_{\geq}^{j-1}} \times \{\infty\} \times \mathbb{Z}_{\geq}^{n-j}}\right)_0$ is isomorphic to the groupoid $\mathfrak{T}_{n-1}|_{\overline{\mathbb{Z}_{\geq}^{j-1}} \times \mathbb{Z}_{\geq}^{n-j}}$ via the groupoid isomorphism

$$(m,k,l,p,\infty,q) \mapsto (m,l,p,q)$$

where

$$(m,k,l,p,\infty,q) \in \mathfrak{G}_n|_{\overline{\mathbb{Z}_{\geq}^{j-1}} \times \{\infty\} \times \mathbb{Z}_{\geq}^{n-j}} \subset \mathbb{Z}^{j-1} \times \mathbb{Z} \times \mathbb{Z}^{n-j} \times \overline{\mathbb{Z}_{\geq}^{j-1}} \times \{\infty\} \times \mathbb{Z}_{\geq}^{n-j}$$

with $\sum_{i=1}^{j-1} m_i + k + \sum_{i=1}^{n-j} l_i = 0$ and hence $k = -\sum m - \sum l$ determined by m, l.

Since $K_1\left(\mathcal{T}^{\otimes j-1} \otimes \mathcal{K}\left(\mathbb{Z}_{\geq}^{n-j}\right)\right) = 0$ and $K_0\left(\mathcal{T}^{\otimes j-1} \otimes \mathcal{K}\left(\mathbb{Z}_{\geq}^{n-j}\right)\right) = \mathbb{Z}$, it is easy to conclude from the cyclic six-term exact sequence of K-groups for the pair $C^*\left(\left(\mathfrak{H}_{j-1}\right)_0\right) \lhd C^*\left(\left(\mathfrak{H}_j\right)_0\right)$ that the following sequence is exact and splits

$$0 \to K_0\left(C^*\left(\left(\mathfrak{H}_{j-1}\right)_0\right)\right) \to K_0\left(C^*\left(\left(\mathfrak{H}_j\right)_0\right)\right) \to K_0\left(\mathcal{T}^{\otimes j-1} \otimes \mathcal{K}\left(\mathbb{Z}_{\geq}^{n-j}\right)\right) \cong \mathbb{Z} \to 0$$

where the projection $\left(\otimes^{j-1} I\right) \otimes \left(\otimes^{n-j} P_1\right)$ is a generator of $K_0\left(\mathcal{T}^{\otimes j-1} \otimes \mathcal{K}\left(\mathbb{Z}_{\geq}^{n-j}\right)\right)$. Note that this $\left(\otimes^{j-1} I\right) \otimes \left(\otimes^{n-j} P_1\right)$ lifts to the projection element $\chi_{A_j} \in C_c\left(\left(\mathfrak{H}_j\right)_0\right) \subset C^*\left(\left(\mathfrak{H}_j\right)_0\right)$ given by the characteristic function of the set

$$A_j := \{0\} \times \{0\} \times \left(\overline{\mathbb{Z}_{\geq}^{j}} \backslash \mathbb{Z}_{\geq}^{j}\right) \times \{0\} \subset \mathfrak{H}_j \subset \mathbb{Z}^j \times \mathbb{Z}^{n-j} \times \overline{\mathbb{Z}_{\geq}^{j}} \times \mathbb{Z}_{\geq}^{n-j}.$$

Furthermore $\chi_{A_j} = Q_{j,\mathrm{id},1}$ in $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right) \subset C\left(\mathbb{S}_H^{2n-1}\right)$. So we get

$$K_0\left(C^*\left(\left(\mathfrak{H}_j\right)_0\right)\right) \cong K_0\left(C^*\left(\left(\mathfrak{H}_{j-1}\right)_0\right)\right) \oplus \mathbb{Z}\left[Q_{j,\Theta,1}\right]$$

with $K_0\left(C^*\left(\left(\mathfrak{H}_{j-1}\right)_0\right)\right)$ canonically embedded in $K_0\left(C^*\left(\left(\mathfrak{H}_j\right)_0\right)\right)$.

Putting together these results for all j, we get

$$K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right) \equiv K_0\left(C^*\left(\left(\mathfrak{H}_n\right)_0\right)\right) \cong \oplus_{j=1}^{n} \mathbb{Z}\left[Q_{j,\mathrm{id},1}\right] \cong \mathbb{Z}^n$$

and hence $Q_{j,\mathrm{id},1}$ freely generate the abelian group $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$. Note that $Q_{j,\mathrm{id},l} = \boxplus^l Q_{j,\mathrm{id},1}$ and $\left[Q_{j,\mathrm{id},l}\right] = l\left[Q_{j,\mathrm{id},1}\right]$ in $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$ for any $l \in \mathbb{N}$.

We now summarize the above discussion.

Theorem 4. The projections $Q_{j,\Theta,l}$ over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ with $l \in \mathbb{N}$ and Θ a $(j, n-j)$-shuffle for $0 < j \leq n$ are mutually inequivalent, and the projections $Q_{j,\mathrm{id},1}$ with $0 < j \leq n$

freely generate the abelian group $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$, such that if $[p]=\sum_{j=1}^{n} m_j\left[Q_{j,\mathrm{id},1}\right]$ for a projection p over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$, then the coefficient m_n of $\left[Q_{n,\mathrm{id},1}\right]$ is the rank of p.

Proof. We only need to note that the rank of $Q_{n,\mathrm{id},1}$ is 1 and the rank of any other $Q_{j,\mathrm{id},1}$ is 0. □

This shows that for $(j',\mathrm{id})\prec(j,\mathrm{id})$ in Ω_0, i.e. $0<j'<j$, it is not true that $Q_{j,\mathrm{id},1}\boxplus Q_{j',\mathrm{id},1}\sim Q_{j,\mathrm{id},1}$ over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ because

$$\left[Q_{j,\mathrm{id},1}\boxplus Q_{j',\mathrm{id},1}\right]=\left[Q_{j,\mathrm{id},1}\right]+\left[Q_{j',\mathrm{id},1}\right]\neq\left[Q_{j,\mathrm{id},1}\right]$$

in $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$.

Next we consider the positive cone of $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$. In the following, we use $\hat{I}$ and $\tilde{I}$ to denote the identity elements of $\mathcal{T}^{\otimes n-1}$ and $\mathcal{T}^{\otimes n}$ respectively.

First, it is easy to see that for $k>0$, the projection $\hat{I}\otimes P_k$ is a sum of k mutually orthogonal projections $\hat{I}\otimes e_{jj}$, each equivalent to $\hat{I}\otimes P_1$ over $\left(\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}\right)^+\subset\mathcal{T}^{\otimes n}$, and hence the projection $\partial_n\left(\hat{I}\otimes P_k\right)$ is a sum of k mutually orthogonal projections $\partial_n\left(\hat{I}\otimes e_{jj}\right)$, each equivalent to $\partial_n\left(\hat{I}\otimes P_1\right)$ over $C\left(\mathbb{S}_H^{2n-1}\right)$. So

$$\hat{I}\otimes P_k\sim\boxplus^k\left(\hat{I}\otimes P_1\right)\equiv\boxplus^k P_{n-1,1}\equiv P_{n-1,k}\ \text{ over }\ \left(\mathcal{T}^{\otimes n-1}\otimes\mathcal{K}\right)^+\subset\mathcal{T}^{\otimes n}$$

and $\partial_n\left(\hat{I}\otimes P_k\right)\sim\boxplus^k Q_{n-1,\mathrm{id},1}\equiv Q_{n-1,\mathrm{id},k}$ over $C\left(\mathbb{S}_H^{2n-1}\right)$. Similarly, by rearranging entries via conjugation by shifts, the projection $\hat{I}\otimes P_{-k}$ is equivalent to $\tilde{I}$ over $\mathcal{T}^{\otimes n}$, and hence $\partial_n\left(\hat{I}\otimes P_{-k}\right)\sim\partial_n\left(\tilde{I}\right)$ over $C\left(\mathbb{S}_H^{2n-1}\right)$. However such equivalences no longer hold over the algebra $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\subset C\left(\mathbb{S}_H^{2n-1}\right)$. For example, $\partial_n\left(\hat{I}\otimes P_{-k}\right)\boxplus\partial_n\left(\hat{I}\otimes P_k\right)\sim\partial_n\left(\tilde{I}\right)$ over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ since $\partial_n\left(\hat{I}\otimes P_{-k}\right)$ and $\partial_n\left(\hat{I}\otimes P_k\right)$ are orthogonal projections in $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$

which add up to $\tilde{I}$. So

$$\left[\partial_n\left(\hat{I}\otimes P_{-1}\right)\right]=\left[\partial_n\left(\tilde{I}\right)\right]-\left[\partial_n\left(\hat{I}\otimes P_1\right)\right]=\left[Q_{n,\mathrm{id},1}\right]-\left[Q_{n-1,\mathrm{id},1}\right]\ \text{ in } K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$$

showing that

$$\left[\partial_n\left(\hat{I}\otimes P_{-1}\right)\right]\in\mathbb{Z}^{n-2}\times\{-1\}\times\{1\}\subset\mathbb{Z}^n\cong K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)$$

and $\partial_n\left(\hat{I}\otimes P_{-1}\right)$ is not even stably equivalent over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$ to any $\boxplus$-sum of the K_0-generating projections $Q_{j,\mathrm{id},1}$ with $0<j\leq n$.

From now on, we include all projections of the form $\partial_n\left(\left(\otimes^{j-1}I\right)\otimes P_k\otimes\left(\otimes^{n-j}P_1\right)\right)$ with $k\in\mathbb{Z}$ as elementary projections over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$, where it is understood that for $k=0$, we take $P_k:=P_{-0}\equiv I$ instead of $P_0\equiv 0$.

Theorem 5. The positive cone of $K_0\left(C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)\right)\cong\mathbb{Z}^n\equiv\bigoplus_{j=1}^n\mathbb{Z}\left[Q_{j,\mathrm{id},1}\right]$ contains

$$\mathbb{Z}^n\backslash\left\{z\in\mathbb{Z}^n : z_j<0=z_{j+1}=\cdots=z_n \text{ for some } 1\leq j\leq n\right\}$$

which is the part of the cone generated/spanned by the equivalence classes of the elementary projections $\partial_n\left(\left(\otimes^{j-1}I\right)\otimes P_k\otimes\left(\otimes^{n-j}P_1\right)\right)$ with $k\in\mathbb{Z}$ and $1\leq j\leq n$, where for $k=0$, we take $P_k:=P_{-0}\equiv I$.

Proof. In [29], it has been established that in the case of $n=2$, the positive cone of

$$K_0\left(C\left(\mathbb{P}^1\left(\mathcal{T}\right)\right)\right)=\mathbb{Z}\left[Q_{1,\mathrm{id},1}\right]\oplus\mathbb{Z}\left[Q_{2,\mathrm{id},1}\right]\equiv\mathbb{Z}\left[\partial_2\left(I\otimes P_1\right)\right]\oplus\mathbb{Z}\left[\partial_2\left(I\otimes I\right)\right]\cong\mathbb{Z}^2$$

consists of $(k,m)\in\mathbb{Z}^2$ with either $k\geq 0$ or the rank $m>0$, such that $\left[\partial_2\left(I\otimes P_k\right)\right]=k\left[\partial_2\left(I\otimes P_1\right)\right]=(k,0)$ and

$$\left[\partial_2\left(I\otimes P_{-k}\right)\right]=\left[\partial_2\left(I\otimes I\right)\right]-k\left[\partial_2\left(I\otimes P_1\right)\right]=(-k,1)$$

in $K_0(C(\mathbb{P}^1(\mathcal{T})))$ for all $k>0$.

By induction on n, we can show that the positive cone of

$$K_0\left(C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\right)=\mathbb{Z}[Q_{1,\mathrm{id},1}]\oplus\cdots\oplus\mathbb{Z}[Q_{n,\mathrm{id},1}]\cong\mathbb{Z}^n$$

contains the set $\left(\mathbb{Z}_{\geq}^{n-1}\times\{0\}\right)\cup\left(\mathbb{Z}^{n-1}\times\mathbb{N}\right)$ consisting of $(k_1,..,k_{n-1},m)\in\mathbb{Z}^n$ with either $k_j\geq 0$ for all j or the rank $m>0$.

Indeed, under the canonical embedding

$$\iota: C\left(\mathbb{P}^{n-2}(\mathcal{T})\right)\equiv C^*\left(\left(\mathfrak{G}_{n-1}\right)_0\right)\to C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\equiv C^*\left(\left(\mathfrak{G}_n\right)_0\right)$$

due to the degree-preserving groupoid embedding of $\left(\mathbb{Z}^{n-1}\ltimes\overline{\mathbb{Z}}^{n-1}\right)\Big|_{\overline{\mathbb{Z}_{\geq}}^{n-1}}$ in $\left(\mathbb{Z}^n\ltimes\overline{\mathbb{Z}}^n\right)\big|_{\overline{\mathbb{Z}_{\geq}}^n}$ as $\left(\left(\mathbb{Z}^{n-1}\times\{0\}\right)\ltimes\left(\overline{\mathbb{Z}}^{n-1}\times\{0\}\right)\right)\Big|_{\overline{\mathbb{Z}_{\geq}}^{n-1}\times\{0\}}$, a projection p (for example, $\partial_{n-1}\left(P_{k_1}\otimes\cdots\otimes P_{k_{n-1}}\right)$) over $C(\mathbb{P}^{n-2}(\mathcal{T}))$ becomes the projection $p\otimes P_1$ (for example, $\partial_n\left(P_{k_1}\otimes\cdots\otimes P_{k_{n-1}}\otimes P_1\right)$) over $C(\mathbb{P}^{n-1}(\mathcal{T}))$. Furthermore if $p\sim q$ over $C(\mathbb{P}^{n-2}(\mathcal{T}))$, say, $upu^{-1}=q$ for some $u\in GL_\infty(C(\mathbb{P}^{n-2}(\mathcal{T})))$ then the equivalence $p\otimes P_1\sim q\otimes P_1$ over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ can be explicitly constructed as

$$\left((u\otimes P_1)+\partial_n(I\otimes P_{-1})\right)(p\otimes P_1)\left((u\otimes P_1)+\partial_n(I\otimes P_{-1})\right)^{-1}=q\otimes P_1$$

with $(u\otimes P_1)+\partial_n(I\otimes P_{-1})\in GL_\infty(C(\mathbb{P}^{n-1}(\mathcal{T})))$. Now consider the well-defined group homomorphism

$$K_0(\iota): K_0\left(C\left(\mathbb{P}^{n-2}(\mathcal{T})\right)\right)\cong\mathbb{Z}^{n-1}\to K_0\left(C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\right)\cong\mathbb{Z}^n$$

mapping the positive cone of $K_0(C(\mathbb{P}^{n-2}(\mathcal{T})))$ into that of $K_0(C(\mathbb{P}^{n-1}(\mathcal{T})))$. Since under ι, the projection $Q_{j,\mathrm{id},1}$ over $C(\mathbb{P}^{n-2}(\mathcal{T}))$ for $0<j\leq n-1$ is sent to the projection $Q_{j,\mathrm{id},1}$ over $C(\mathbb{P}^{n-1}(\mathcal{T}))$, by induction hypothesis, we get that the positive cone

of $K_0(C(\mathbb{P}^{n-1}(\mathcal{T}))) \cong \mathbb{Z}^n$ contains $\left(\mathbb{Z}_{\geq}^{n-2} \times \{0\} \times \{0\}\right) \cup (\mathbb{Z}^{n-2} \times \mathbb{N} \times \{0\})$, and hence $\left(\mathbb{Z}_{\geq}^{n-1} \times \{0\}\right) \cup (\mathbb{Z}^{n-2} \times \mathbb{N} \times \mathbb{Z}_{\geq})$.

On the other hand, for $k > 0$,

$$\hat{I} \otimes P_{-k} = \left(\hat{I} \otimes P_{-(k+1)}\right) \boxplus \left(\hat{I} \otimes e_{kk}\right) \sim \left(\hat{I} \otimes P_{-(k+1)}\right) \boxplus \left(I' \otimes P_{-(k-1)} \otimes P_1\right) \text{ over } \mathcal{T}^{\otimes n}$$

where I' denotes the identity element of $\mathcal{T}^{\otimes n-2}$, because $\hat{I} \otimes P_{-k}$ is the sum of orthogonal projections $\left(\hat{I} \otimes P_{-(k+1)}\right)$ and $\left(\hat{I} \otimes e_{kk}\right)$, and $\left(\hat{I} \otimes e_{kk}\right) \boxplus 0$ when conjugated by

$$u_k := \begin{pmatrix} I' \otimes I \otimes P_{k-1} & I' \otimes \left(\mathcal{S}^{k-1}\right)^* \otimes \mathcal{S}^{k-1} \\ I' \otimes \mathcal{S}^{k-1} \otimes \left(\mathcal{S}^{k-1}\right)^* & I' \otimes P_{k-1} \otimes I \end{pmatrix} \in GL_2\left(\mathcal{T}^{\otimes n}\right)$$

becomes $0 \boxplus \left(I' \otimes P_{-(k-1)} \otimes P_1\right)$. Since $\partial_n(u_k)$ of total degree 0 is in $M_2(C(\mathbb{P}^{n-1}(\mathcal{T})))$, we get

$$\partial_n\left(\hat{I} \otimes P_{-k}\right) \sim \partial_n\left(\left(\hat{I} \otimes P_{-(k+1)}\right)\right) \boxplus \iota\left(\partial_{n-1}\left(I' \otimes P_{-(k-1)}\right)\right) \text{ over } C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)$$

and hence

$$\left[\partial_n\left(\hat{I} \otimes P_{-k}\right)\right] - \left[\partial_n\left(\left(\hat{I} \otimes P_{-(k+1)}\right)\right)\right] \in \mathbb{Z}^{n-2} \times \{1\} \times \{0\} \text{ in } \mathbb{Z}^n$$

because $\left[\partial_{n-1}\left(I' \otimes P_{-(k-1)}\right)\right] \in \mathbb{Z}^{n-2} \times \{1\}$ for the rank-one projection $I' \otimes P_{-(k-1)}$ over $\mathcal{T}^{\otimes n-1}$. With

$$\left[\partial_n\left(\hat{I} \otimes P_{-1}\right)\right] = \left[\partial_n\left(\tilde{I}\right)\right] - \left[\partial_n\left(\hat{I} \otimes P_1\right)\right] = (0, ..., 0, -1, 1) \in \mathbb{Z}^n,$$

we get inductively

$$\left[\partial_n\left(\hat{I} \otimes P_{-k}\right)\right] \in \mathbb{Z}^{n-2} \times \{-k\} \times \{1\} \subset \mathbb{Z}^n \cong K_0\left(C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)\right)$$

for all $k > 0$. Thus the positive cone of $K_0(C(\mathbb{P}^{n-1}(\mathcal{T}))) \cong \mathbb{Z}^n$ contains $\left(\mathbb{Z}_{\geq}^{n-1} \times \{0\}\right) \cup (\mathbb{Z}^{n-1} \times \{1\})$ and hence $\left(\mathbb{Z}_{\geq}^{n-1} \times \{0\}\right) \cup (\mathbb{Z}^{n-1} \times \mathbb{N})$. On the other hand, the positive cone of $K_0(C(\mathbb{P}^{n-2}(\mathcal{T})))$ is mapped into the positive cone of $K_0(C(\mathbb{P}^{n-1}(\mathcal{T})))$ by the homomorphism $\cdot \times \{0\} \equiv K_0(\iota)$. So it is easy to get inductively the conclusion. □

We note that for the case of $n = 2$, the finitely generated projective modules over $C(\mathbb{P}^1(\mathcal{T}))$ are completely classified with the positive cone of $K_0(C(\mathbb{P}^1(\mathcal{T})))$ explicitly identified in [29].

8 Quantum line bundles

In this section, we identify the quantum line bundles $L_k := C\left(\mathbb{S}_H^{2n-1}\right)_k$ of degree k over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ with a concrete (equivalence class of) projection described in terms of the elementary projections defined in the previous section. We continue to use $\hat{I}$ and $\tilde{I}$ to denote the identity elements of $\mathcal{T}^{\otimes n-1}$ and $\mathcal{T}^{\otimes n}$ respectively, and we start to use $0^{(l)}$ to denote the zero of $\mathbb{Z}^l$

To distinguish between ordinary function product and convolution product, we denote the groupoid C*-algebraic (convolution) multiplication of elements in $C_c(\mathcal{G}) \subset C^*(\mathcal{G})$ by $*$, while omitting $*$ when the elements are presented as operators or when they are multiplied together pointwise as functions on $\mathcal{G}$. We also view $C_c(\mathfrak{G}_n)$ or $C_c((\mathfrak{G}_n)_k)$ (also abbreviated as $C_c(\mathfrak{G}_n)_k$) as left $C_c(\mathfrak{G}_n)_0$-modules with $C_c(\mathfrak{G}_n)$ carrying the convolution algebra structure as a subalgebra of the groupoid C*-algebra $C^*(\mathfrak{G}_n)$. Similarly, for a closed subset X of the unit space of $\mathfrak{G}_n$, the inverse image $\mathfrak{G}_n \restriction_X$of X under the source map of $\mathfrak{G}_n$ or its grade-k

component $(\mathfrak{G}_n\restriction_X)_k$ gives rise to a left $C_c(\mathfrak{G}_n)_0$-module $C_c(\mathfrak{G}_n\restriction_X)$ or $C_c(\mathfrak{G}_n\restriction_X)_k$.

We define a partial isometry in $C\left(\mathbb{S}_H^{2n-1}\right)\equiv C^*(\mathfrak{G}_n)$ for each $k\in\mathbb{Z}$ as the characteristic function χ_{B_k} of the compact open set

$$B_k:=\left\{(0,k,p,q)\in\mathfrak{G}_n\subset\mathbb{Z}^{n-1}\times\mathbb{Z}\times\overline{\mathbb{Z}_{\geq}^{n-1}}\times\overline{\mathbb{Z}_{\geq}}:q+k\geq 0\right\}\subset\mathfrak{G}_n.$$

It is easy to verify that $\chi_{B_k}\in C_c(\mathfrak{G}_n)_k$ and $(\chi_{B_k})^*\in C_c(\mathfrak{G}_n)_{-k}$ such that

$$(\chi_{B_k})^**\chi_{B_k}=\begin{cases}\chi_{\{0^{(n)}\}\times\left(\overline{\mathbb{Z}_{\geq}^n}\setminus\mathbb{Z}_{\geq}^n\right)}=1_{C^*(\mathfrak{G}_n)}\equiv 1_{C^*(\mathfrak{G}_n)_0} & \text{if}\quad k\geq 0\\ \chi_{\{0^{(n)}\}\times\left(\left(\overline{\mathbb{Z}_{\geq}^{n-1}}\times\overline{\mathbb{Z}_{\geq|k|}}\right)\setminus\mathbb{Z}_{\geq}^n\right)}=\partial_n\left(\hat{I}\otimes P_{-|k|}\right) & \text{if}\quad k<0\end{cases}$$

and

$$\chi_{B_k}*(\chi_{B_k})^*=\begin{cases}\chi_{\{0^{(n)}\}\times\left(\left(\overline{\mathbb{Z}_{\geq}^{n-1}}\times\overline{\mathbb{Z}_{\geq k}}\right)\setminus\mathbb{Z}_{\geq}^n\right)}=\partial_n\left(\hat{I}\otimes P_{-k}\right) & \text{if}\quad k\geq 0\\ \chi_{\{0^{(n)}\}\times\left(\overline{\mathbb{Z}_{\geq}^n}\setminus\mathbb{Z}_{\geq}^n\right)}=1_{C^*(\mathfrak{G}_n)}\equiv 1_{C^*(\mathfrak{G}_n)_0} & \text{if}\quad k<0\end{cases}.$$

For $k\geq 0$, we have $C_c(\mathfrak{G}_n)_k*(\chi_{B_k})^*\subset C_c(\mathfrak{G}_n)_0$ and

$$\left(C_c(\mathfrak{G}_n)_k*(\chi_{B_k})^*\right)*\chi_{B_k}=C_c(\mathfrak{G}_n)_k$$

which implies that the convolution operator $\cdot*\chi_{B_k}$ maps $C_c(\mathfrak{G}_n)_0$ onto $C_c(\mathfrak{G}_n)_k$. Since $\chi_{B_k}*(\chi_{B_k})^*=\chi_{\{0^{(n)}\}\times\left(\left(\overline{\mathbb{Z}_{\geq}^{n-1}}\times\overline{\mathbb{Z}_{\geq k}}\right)\setminus\mathbb{Z}_{\geq}^n\right)}$, we get $\cdot*\chi_{B_k}$ mapping $C_c(\mathfrak{G}_n)_0*\chi_{\{0^{(n)}\}\times\left(\left(\overline{\mathbb{Z}_{\geq}^{n-1}}\times\overline{\mathbb{Z}_{\geq k}}\right)\setminus\mathbb{Z}_{\geq}^n\right)}$ bijectively onto $C_c(\mathfrak{G}_n)_k$ with $\cdot*(\chi_{B_k})^*$ as the inverse. Furthermore $\cdot*\chi_{B_k}$ is a left $C_c(\mathfrak{G}_n)_0$-module homomorphism. With χ_{B_k} being a partial isometry, $\cdot*\chi_{B_k}$ and $\cdot*(\chi_{B_k})^*$ extend continuously to provide an isomorphism between the $C^*(\mathfrak{G}_n)_0$-modules

$$C^*(\mathfrak{G}_n)_0\,\chi_{\{0^{(n)}\}\times\left(\left(\overline{\mathbb{Z}_{\geq}^{n-1}}\times\overline{\mathbb{Z}_{\geq k}}\right)\setminus\mathbb{Z}_{\geq}^n\right)}\equiv C^*(\mathfrak{G}_n)_0\,\partial_n\left(\hat{I}\otimes P_{-k}\right)$$

and $C^*(\mathfrak{G}_n)_k\equiv\overline{C_c(\mathfrak{G}_n)_k}$. So the quantum line bundle $C^*(\mathfrak{G}_n)_k$ is identified with the projection $\partial_n\left(\hat{I}\otimes P_{-k}\right)$.

For $k<0$, we consider the direct sum decomposition as left $C_c(\mathfrak{G}_n)_0$-modules

$$C_c(\mathfrak{G}_n)_k = \left(C_c(\mathfrak{G}_n)_k * \chi_{\left\{0^{(n)}\right\}\times\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0,1,\dots,|k|-1\}}\right) \oplus \left(C_c(\mathfrak{G}_n)_k * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}_\geq^{n-1}}\times\overline{\mathbb{Z}_{\geq|k|}}\right)\setminus\mathbb{Z}_\geq^{n}\right)}\right)$$
$$= C_c\left(\mathfrak{G}_n|_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0,1,\dots,|k|-1\}}\right)_k \oplus \left(C_c(\mathfrak{G}_n)_k * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}_\geq^{n-1}}\times\overline{\mathbb{Z}_{\geq|k|}}\right)\setminus\mathbb{Z}_\geq^{n}\right)}\right).$$

From

$$C_c(\mathfrak{G}_n)_0 * \chi_{B_k} * (\chi_{B_k})^* \equiv C_c(\mathfrak{G}_n)_0 * 1_{C^*(\mathfrak{G}_n)} = C_c(\mathfrak{G}_n)_0$$

and

$$C_c(\mathfrak{G}_n)_k * (\chi_{B_k})^* * \chi_{B_k} = C_c(\mathfrak{G}_n)_k * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}_\geq^{n-1}}\times\overline{\mathbb{Z}_{\geq|k|}}\right)\setminus\mathbb{Z}_\geq^{n}\right)}$$

we see that $\cdot * \chi_{B_{|k|}}$ is a left $C_c(\mathfrak{G}_n)_0$-module isomorphism between $C_c(\mathfrak{G}_n)_0$ and $C_c(\mathfrak{G}_n)_k * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}_\geq^{n-1}}\times\overline{\mathbb{Z}_{\geq|k|}}\right)\setminus\mathbb{Z}_\geq^{n}\right)}$ with $\cdot * (\chi_{B_k})^*$ as its inverse.

On the other hand, in the $C_c(\mathfrak{G}_n)_0$-module decomposition

$$C_c\left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0,1,\dots,|k|-1\}}\right)_k = \bigoplus_{j=0}^{|k|-1} C_c\left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{j\}}\right)_k,$$

each $C_c\left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{j\}}\right)_k$ is isomorphic to the $C_c(\mathfrak{G}_n)_0$-module $C_c\left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0\}}\right)_{k+j}$ with $k+j<0$ via the homeomorphism

$$(m,l,p,j) \in \left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{j\}}\right)_k \subset \mathbb{Z}^{n-1}\times\mathbb{Z}\times\overline{\mathbb{Z}_\geq^{n-1}}\times\mathbb{Z}_\geq \mapsto (m,l+j,p,0) \in \left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0\}}\right)_{k+j}$$

where the implicit condition $l+j\geq 0$ is equivalent to $l\geq -j$. So we focus on analyzing $C_c(\mathfrak{G}_n)_0$-modules of the form

$$C_c\left(\left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0\}}\right)_{-r}\right) = C_c(\mathfrak{G}_n)_{-r} * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}_\geq^{n-1}}\setminus\mathbb{Z}_\geq^{n-1}\right)\times\{0\}\right)} = C_c(\mathfrak{G}_n)_{-r}\,\partial_n\left(\hat{I}\otimes P_1\right)$$

with $r \geq 0$. Note that the $C^*\left(\mathfrak{G}_n\right)_0$-module

$$\overline{C_c\left(\mathfrak{G}_n\right)_0 \partial_n\left(\hat{I} \otimes P_1\right)} = C^*\left(\mathfrak{G}_n\right)_0 \partial_n\left(\hat{I} \otimes P_1\right)$$

is identified with the projection $\partial_n\left(\hat{I} \otimes P_1\right) \equiv Q_{n-1,\mathrm{id},1}$.

For $r > 0$, similar to the argument used above, it can be checked that the compact open subset

$$B'_{-r} := \left\{(0,-r,0,p,q,0) \in \mathfrak{G}_n \subset \mathbb{Z}^{n-2} \times \mathbb{Z} \times \mathbb{Z} \times \overline{\mathbb{Z}^{n-2}_{\geq}} \times \overline{\mathbb{Z}_{\geq}} \times \overline{\mathbb{Z}_{\geq}} : q \geq r\right\} \subset \mathfrak{G}_n$$

defines a partial isometry $\chi_{B'_{-r}} \in C_c\left(\mathfrak{G}_n\right)_{-r}$ with $\left(\chi_{B'_{-r}}\right)^* \in C_c\left(\mathfrak{G}_n\right)_r$ such that

$$\left(\chi_{B'_{-r}}\right)^* * \chi_{B'_{-r}} = \chi_{\left\{0^{(n)}\right\} \times \left(\left(\overline{\mathbb{Z}^{n-2}_{\geq}} \times \overline{\mathbb{Z}_{\geq r}} \times \{0\}\right) \setminus \mathbb{Z}^n_{\geq}\right)} = \partial_n\left(I^{\otimes n-2} \otimes P_{-r} \otimes I\right)$$

and

$$\chi_{B'_{-r}} * \left(\chi_{B'_{-r}}\right)^* = \chi_{\left\{0^{(n)}\right\} \times \left(\overline{\mathbb{Z}^{n-1}_{\geq}} \setminus \mathbb{Z}^{n-1}_{\geq}\right) \times \{0\}}.$$

In the decomposition

$$\begin{aligned}
&C_c\left(\mathfrak{G}_n\right)_{-r} * \chi_{\left\{0^{(n)}\right\} \times \left(\left(\overline{\mathbb{Z}^{n-1}_{\geq}} \setminus \mathbb{Z}^{n-1}_{\geq}\right) \times \{0\}\right)} \\
&= \left(C_c\left(\mathfrak{G}_n\right)_{-r} * \left(\chi_{\left\{0^{(n)}\right\} \times \left(\overline{\mathbb{Z}^{n-2}_{\geq}} \setminus \mathbb{Z}^{n-2}_{\geq}\right) \times \{0,1,\ldots,r-1\} \times \{0\}}\right)\right) \oplus \left(C_c\left(\mathfrak{G}_n\right)_{-r} * \chi_{\left\{0^{(n)}\right\} \times \left(\left(\overline{\mathbb{Z}^{n-2}_{\geq}} \times \overline{\mathbb{Z}_{\geq r}} \times \{0\}\right) \setminus \mathbb{Z}^n_{\geq}\right)}\right) \\
&= C_c\left(\mathfrak{G}_n \restriction_{\left(\overline{\mathbb{Z}^{n-2}_{\geq}} \setminus \mathbb{Z}^{n-2}_{\geq}\right) \times \{0,1,\ldots,r-1\} \times \{0\}}\right)_{-r} \oplus \left(C_c\left(\mathfrak{G}_n\right)_{-r} * \chi_{\left\{0^{(n)}\right\} \times \left(\left(\overline{\mathbb{Z}^{n-2}_{\geq}} \times \overline{\mathbb{Z}_{\geq r}} \times \{0\}\right) \setminus \mathbb{Z}^n_{\geq}\right)}\right),
\end{aligned}$$

the second summand is isomorphic, via the right convolution $\cdot * \left(\chi_{B'_{-r}}\right)^*$ by the partial isometry $\left(\chi_{B'_{-r}}\right)^*$, to the $C_c\left(\mathfrak{G}_n\right)_0$-module

$$C_c\left(\mathfrak{G}_n\right)_0 * \chi_{\left\{0^{(n)}\right\} \times \left(\left(\overline{\mathbb{Z}^{n-1}_{\geq}} \times \{0\}\right) \setminus \mathbb{Z}^n_{\geq}\right)} = C_c\left(\mathfrak{G}_n\right)_0 \partial_n\left(\hat{I} \otimes P_1\right).$$

Now we introduce the notation of a $C_c(\mathfrak{G}_n)_0$-module

$$A_{r,l} := C_c\left(\left(\mathfrak{G}_n\restriction_{\left(\overline{\mathbb{Z}^l_{\geq}}\setminus\mathbb{Z}^l_{\geq}\right)\times\left\{0^{(n-l)}\right\}}\right)_{-r}\right) \subset C_c\left((\mathfrak{G}_n)_{-r}\right) \subset C_c(\mathfrak{G}_n)$$

for $r \geq 0$ and $1 \leq l \leq n-1$. We note that the $C_c(\mathfrak{G}_n)_0$-module

$$A_{r,1} = C_c\left(\left(\mathfrak{G}_n\restriction_{\{\infty\}\times\left\{0^{(n-1)}\right\}}\right)_{-r}\right)$$

is isomorphic to

$$C_c\left(\left(\mathfrak{G}_n\restriction_{\{\infty\}\times\left\{0^{(n-1)}\right\}}\right)_0\right) = C_c\left(\left(\mathfrak{G}_n\restriction_{\left(\overline{\mathbb{Z}_{\geq}}\times\left\{0^{(n-1)}\right\}\right)\setminus\mathbb{Z}^n_{\geq}}\right)_0\right)$$

$$\equiv C_c(\mathfrak{G}_n)_0 * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}_{\geq}}\times\left\{0^{(n-1)}\right\}\right)\setminus\mathbb{Z}^n_{\geq}\right)} = C_c(\mathfrak{G}_n)_0\,\partial_n\left(I\otimes P_1^{\otimes n-1}\right)$$

via the homeomorphism

$$\left(s,t,\infty,0^{(n-1)}\right) \in \left(\mathfrak{G}_n\restriction_{\{\infty\}\times\left\{0^{(n-1)}\right\}}\right)_{-r} \subset \mathbb{Z}\times\mathbb{Z}^{n-1}\times\{\infty\}\times\overline{\mathbb{Z}^{n-1}_{\geq}}$$
$$\mapsto \left(s+r,t,\infty,0^{(n-1)}\right) \in \left(\mathfrak{G}_n\restriction_{\{\infty\}\times\left\{0^{(n-1)}\right\}}\right)_0.$$

Applying the same kind of arguments as shown above, we get the isomorphism of $C_c(\mathfrak{G}_n)_0$-modules

$$\begin{aligned}
A_{r,l} &\cong C_c\left(\mathfrak{G}_n\restriction_{\left(\overline{\mathbb{Z}^{l-1}_{\geq}}\setminus\mathbb{Z}^{l-1}_{\geq}\right)\times\{0,1,\ldots,r-1\}\times\left\{0^{(n-l)}\right\}}\right)_{-r} \oplus \left(C_c(\mathfrak{G}_n)_0 * \chi_{\left\{0^{(n)}\right\}\times\left(\left(\overline{\mathbb{Z}^l_{\geq}}\setminus\mathbb{Z}^l_{\geq}\right)\times\left\{0^{(n-l)}\right\}\right)}\right)\\
&\cong \bigoplus_{j=0}^{r-1} C_c\left(\mathfrak{G}_n\restriction_{\left(\overline{\mathbb{Z}^{l-1}_{\geq}}\setminus\mathbb{Z}^{l-1}_{\geq}\right)\times\{j\}\times\left\{0^{(n-l)}\right\}}\right)_{-r} \oplus \left(C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes l}\otimes P_1^{\otimes n-l}\right)\right)\\
&\cong \bigoplus_{j=0}^{r-1} C_c\left(\mathfrak{G}_n\restriction_{\left(\overline{\mathbb{Z}^{l-1}_{\geq}}\setminus\mathbb{Z}^{l-1}_{\geq}\right)\times\{0\}\times\left\{0^{(n-l)}\right\}}\right)_{-r+j} \oplus \left(C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes l}\otimes P_1^{\otimes n-l}\right)\right)\\
&= \bigoplus_{j=0}^{r-1} A_{r-j,l-1} \oplus \left(C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes l}\otimes P_1^{\otimes n-l}\right)\right)
\end{aligned}$$

for $2 \leq l \leq n-1$. This provides a recursive formula to reduce the index l of the module $A_{r,l}$.

For $n > 2$ fixed, we define a combinatorial number $\nu_n(k,m,l)$ recursively by

$$\nu_n(k,m,l) := \sum_{s=0}^{m} \nu_n(k,s,l-1)$$

and $\nu_n(k,m,1) := 1$, for $k > m \geq 0$ and $2 \leq l \leq n-1$, to be used in the following theorem.

Theorem 6. For $n > 2$, the quantum line bundle $L_k \equiv C\left(\mathbb{S}_H^{2n-1}\right)_k$ of degree $k \in \mathbb{Z}$ over $C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)$ is isomorphic to the finitely generated projective left module over $C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)$ determined by the projection $\partial_n\left(\otimes^{n-1} I \otimes P_{-k}\right)$ if $k \geq 0$, and the projection

$$\left(\boxplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(|k|,m,n-2)} \partial_n\left(I \otimes P_1^{\otimes n-1}\right)\right) \boxplus \left(\boxplus_{l=1}^{n-1} \boxplus^{\nu_n(|k|,|k|-1,l)} \partial_n\left(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\right)\right)$$

if $k < 0$.

Proof. Only the case of $k < 0$ remains to be proved as follows.

For $k < 0$, starting with the established isomorphism

$$C_c(\mathfrak{G}_n)_k = C_c\left(\mathfrak{G}_n \upharpoonright_{\left(\overline{\mathbb{Z}_{\geq}^{n-1}} \setminus \mathbb{Z}_{\geq}^{n-1}\right) \times \{0,1,\ldots,|k|-1\}}\right)_k \oplus C_c(\mathfrak{G}_n)_0$$

$$= \bigoplus_{m=0}^{|k|-1} C_c\left(\mathfrak{G}_n \upharpoonright_{\left(\overline{\mathbb{Z}_{\geq}^{n-1}} \setminus \mathbb{Z}_{\geq}^{n-1}\right) \times \{m\}}\right)_k \oplus C_c(\mathfrak{G}_n)_0 \cong \bigoplus_{m=0}^{|k|-1} A_{|k|-m,n-1} \oplus C_c(\mathfrak{G}_n)_0,$$

we apply repeatedly the recursive formula

$$A_{r,l} = \bigoplus_{j=0}^{r-1} A_{r-j,l-1} \oplus \left(C_c(\mathfrak{G}_n)_0 \, \partial_n\left(I^{\otimes l} \otimes P_1^{\otimes n-l}\right)\right)$$

reducing l for $A_{r,l}$ with $2 \leq l \leq n$ until l reaches 2 with

$$A_{r,2} \cong \left(\bigoplus_{j=0}^{r-1} \left(C_c(\mathfrak{G}_n)_0 \, \partial_n\left(I \otimes P_1^{\otimes n-1}\right)\right)\right) \oplus \left(C_c(\mathfrak{G}_n)_0 \, \partial_n\left(I^{\otimes 2} \otimes P_1^{\otimes n-2}\right)\right),$$

in order to convert all terms to $C_c(\mathfrak{G}_n)_0$-modules of the form $C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes j}\otimes P_1^{\otimes n-j}\right)$ for some $0<j\leq n$.

In fact, we check inductively on $1\leq j\leq n-2$ that

$$(*)\quad C_c(\mathfrak{G}_n)_k \cong \bigoplus_{m=0}^{|k|-1}\left(\oplus^{\nu_n(|k|,m,j)}A_{|k|-m,n-j}\right)\oplus\bigoplus_{l=1}^{j}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right).$$

The case of $j=1$ is our starting point already proved. Now assuming that it holds for j, we get by the above recursive formula

$$\begin{aligned}
C_c(\mathfrak{G}_n)_k &\cong \bigoplus_{m=0}^{|k|-1}\oplus^{\nu_n(|k|,m,j)}\left(\left(\bigoplus_{s=0}^{|k|-m-1}A_{|k|-m-s,n-j-1}\right)\oplus\left(C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-j}\otimes P_1^{\otimes j}\right)\right)\right)\\
&\quad\oplus\left(\bigoplus_{l=1}^{j}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\cong\left(\bigoplus_{m=0}^{|k|-1}\oplus^{\nu_n(|k|,m,j)}\bigoplus_{s=0}^{|k|-m-1}A_{|k|-m-s,n-j-1}\right)\oplus\left(\oplus^{\sum_{m=0}^{|k|-1}\nu_n(|k|,m,j)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-j}\otimes P_1^{\otimes j}\right)\right)\\
&\oplus\left(\bigoplus_{l=1}^{j}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right)\right)\\
&\cong\left(\bigoplus_{m=0}^{|k|-1}\oplus^{\nu_n(|k|,m,j)}\bigoplus_{s=0}^{|k|-m-1}A_{|k|-m-s,n-j-1}\right)\oplus\bigoplus_{l=1}^{j+1}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right)\\
&=\left(\bigoplus_{m=0}^{|k|-1}\bigoplus_{s=0}^{m}\oplus^{\nu_n(|k|,s,j)}A_{|k|-m,n-j-1}\right)\oplus\bigoplus_{l=1}^{j+1}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right)\\
&=\left(\bigoplus_{m=0}^{|k|-1}\oplus^{\nu_n(|k|,m,j+1)}A_{|k|-m,n-j-1}\right)\oplus\bigoplus_{l=1}^{j+1}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right),
\end{aligned}$$

which verifies (*) for $j+1$.

For $j=n-2$, (*) says

$$C_c(\mathfrak{G}_n)_k \cong \bigoplus_{m=0}^{|k|-1}\left(\oplus^{\nu_n(|k|,m,n-2)}A_{|k|-m,2}\right)\oplus\bigoplus_{l=1}^{n-2}\left(\oplus^{\nu_n(|k|,|k|-1,l)}C_c(\mathfrak{G}_n)_0\,\partial_n\left(I^{\otimes n-l+1}\otimes P_1^{\otimes l-1}\right)\right)$$

$$\cong \bigoplus_{m=0}^{|k|-1} \oplus^{\nu_n(|k|,m,n-2)} \left(\left(\bigoplus_{j=0}^{|k|-m-1} \left(C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I \otimes P_1^{\otimes n-1}\right)\right)\right) \oplus \left(C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I^{\otimes 2} \otimes P_1^{\otimes n-2}\right)\right)\right)$$

$$\oplus \bigoplus_{l=1}^{n-2} \left(\oplus^{\nu_n(|k|,|k|-1,l)} C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\right)\right)$$

$$\cong \left(\oplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(|k|,m,n-2)} C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I \otimes P_1^{\otimes n-1}\right)\right)$$
$$\oplus \left(\oplus^{\sum_{m=0}^{|k|-1}\nu_n(|k|,m,n-2)} \left(C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I^{\otimes 2} \otimes P_1^{\otimes n-2}\right)\right)\right)$$
$$\oplus \bigoplus_{l=1}^{n-2} \left(\oplus^{\nu_n(|k|,|k|-1,l)} C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\right)\right)$$

$$\cong \left(\oplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(|k|,m,n-2)} C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I \otimes P_1^{\otimes n-1}\right)\right)$$
$$\oplus \bigoplus_{l=1}^{n-1} \left(\oplus^{\nu_n(|k|,|k|-1,l)} C_c\left(\mathfrak{G}_n\right)_0 \partial_n \left(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\right)\right).$$

After completion, we get the $C^*\left(\left(\mathfrak{G}_n\right)_0\right)$-module L_k isomorphic to

$$\left(\oplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(|k|,m,n-2)} C^*\left(\left(\mathfrak{G}_n\right)_0\right) \partial_n \left(I \otimes P_1^{\otimes n-1}\right)\right)$$
$$\oplus \bigoplus_{l=1}^{n-1} \left(\oplus^{\nu_n(|k|,|k|-1,l)} C^*\left(\left(\mathfrak{G}_n\right)_0\right) \partial_n \left(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\right)\right)$$

which corresponds to the projection

$$\left(\boxplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(|k|,m,n-2)} \partial_n \left(I \otimes P_1^{\otimes n-1}\right)\right) \boxplus \left(\boxplus_{l=1}^{n-1} \left(\boxplus^{\nu_n(|k|,|k|-1,l)} \partial_n \left(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\right)\right)\right).$$

□

Little is known about the cancellation problem and hence the classification problem for finitely generated projective modules over $C\left(\mathbb{P}^{n-1}\left(\mathcal{T}\right)\right)$. We expect that these problems will be far more complicated than those for over $C\left(\mathbb{S}_H^{2n-1}\right)$ and $C\left(\mathcal{T}^{\otimes n}\right)$.

References

[1] K. A. Bach, *A cancellation problem for quantum spheres*, Thesis, U. of Kansas, Lawrence, 2003.

[2] P. Baum, P. Hajac, R. Matthes, and W. Szymański, *The K-theory of Heegaard-type quantum 3-sphere,* K-Theory 35 (2005), no. 1-2, 159–186.

[3] B. Blackadar, "*K-theory for Operator Algebras*", MSRI Publications Vol. 5,. Cambridge University Press, Cambridge, 2nd ed., 1998.

[4] T. Brzeziński and S. A. Fairfax, *Quantum teardrops*, Comm. Math. Phys. 316 (2012), 151-170.

[5] A. Connes, "*Noncommutative Geometry*", Academic Press, New York, 1994.

[6] R. E. Curto and P. S. Muhly, *C*-algebras of multiplication operators on Bergman spaces*, J. Func. Anal. 64 (1985), 315-329.

[7] C. Farsi, P. M. Hajac, T. Maszczyk, and B. Zieliński, *Rank-two Milnor idempotents for the multipullback quantum complex projective plane*, arXiv:1708.04426.

[8] P. M. Hajac, *Strong connections on quantum principal bundles*, Comm. Math. Phys. 182 (1996), 579–617.

[9] P. M. Hajac, A. Kaygun, and B. Zieliński, *Quantum complex projective spaces from Toeplitz cubes*, J. Noncomm. Geom. 6 (2012), 603-621.

[10] P. M. Hajac, R. Matthes, and W. Szymański, *Chern numbers for two families of noncommutative Hopf fibrations*, C. R. Acad. Sci. Paris, Ser. I 336 (2003), 925-930.

[11] P. M. Hajac, R. Matthes, and W. Szymański, *Noncommutative index theory for mirror quantum spheres*, C. R. Acad. Sci. Paris, Ser. I 343 (2006), 731-736.

[12] P. Hajac, R. Nest, D. Pask, A. Sims, and B. Zieliński, *The K-theory of twisted multipullback quantum odd spheres and complex projective spaces*, arXiv:1512.08816v2.

[13] R. H. Herman and L. N. Vaserstein, *The stable range of C*-algebras*, Invent. Math. 77 (1984), 553-555.

[14] U. Meyer, *Projective quantum spaces*, Lett. Math. Phys. 35 (1995), 91–97.

[15] P. S. Muhly and J. N. Renault, *C*-algebras of multivariable Wiener-Hopf operators*, Trans. Amer. Math. Soc. 274 (1982), 1-44.

[16] G. Nagy, *Stable rank of C*-algebras of Toeplitz operators on polydisks*, Operators in Indefinite Metric Spaces, Scattering Theory and Other Topics (Bucharest, 1985), 227–235, Oper.Theory Adv.Appl., 24, Birkhäuser, Basel, 1987.

[17] V. Nistor, *Stable range for tensor products of extensions of $\mathcal{K}$ by $C(X)$*, J. Operator Theory, 16 (1986), no. 2, 387–396.

[18] G. K. Pedersen, *Pullback and pushout constructions in C*-algebra theory*, J. Func. Anal. 267 (1999), 243-344.

[19] M. A. Peterka, *Finitely-generated projective modules over the θ-deformed 4-sphere*, Comm. Math. Phys. 321, 577–603 (2013).

[20] J. Renault, "*A Groupoid Approach to C*-algebras*", Lecture Notes in Mathematics, Vol. 793, Springer-Verlag, New York, 1980.

[21] M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. 46 (1983), 301-333.

[22] ——, *The cancellation theorem for projective modules over irrational rotation C*-algebras*, Proc. London Math. Soc. 47 (1983), 285-302.

[23] ——, *Projective modules over higher-dimensional noncommutative tori*, Canad. J. Math., 40 (1988), 257–338.

[24] N. Salinas, A. J. L. Sheu, and H. Upmeier, *Toeplitz operators on pseudoconvex domains and foliation C*-algebras*, Ann. Math., 130 (1989), 531-565.

[25] A. J. L. Sheu, *A cancellation theorem for modules over the group C*-algebras of certain nilpotent Lie groups*, Canad. J. Math., 39 (1987), 365-427.

[26] ——, *Compact quantum groups and groupoid C*-algebras*, J. Func. Anal. 144 (1997), 371-393.

[27] ——, *Quantization of the Poisson SU(2) and its Poisson homogeneous space - the 2-sphere*, Comm. Math. Phys. 135 (1991), 217-232.

[28] ——, *The structure of line bundles over quantum teardrops*, SIGMA 10 (2014), 027.

[29] ______, *Projective modules over quantum projective line*, International J. Math., 28 (2017), 1750022.

[30] R. W. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. 705 (1962), 264-277.

[31] J. Taylor, *Banach algebras and topology*, in “Algebras in Analysis”, Academic Press, New York, 1975.

[32] L. L. Vaksman and Ya. S. Soibelman, *The algebra of functions on the quantum group* $SU(n+1)$*, and odd-dimensional quantum spheres*, Leningrad Math. J. 2 (1991), 1023-1042.

[33] G. W. Whitehead, “*Elements of Homotopy Theory*”, Springer-Verlag, New York, 1978.

[34] S. L. Woronowicz, *Compact quantum groups*, in “Les Houches, Session LXIV, 1995, Quantum Symmetries”, Elsevier, 1998, pp. 845-884.

[35] ______, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), 613-665.