Projective Modules over Quantum Projective Line*

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Abstract

Taking a groupoid C*-algebra approach to the study of the quantum complex projective spaces $\mathbb{P}^n(\mathcal{T})$ constructed from the multipullback quantum spheres introduced by Hajac and collaborators, we analyze the structure of the C*-algebra $C\left(\mathbb{P}^1(\mathcal{T})\right)$ realized as a concrete groupoid C*-algebra, and find its K-groups. Furthermore after a complete classification of the unitary equivalence classes of projections or equivalently the isomorphism classes of finitely generated projective modules over the C*-algebra $C\left(\mathbb{P}^1(\mathcal{T})\right)$, we identify those quantum principal $U(1)$-bundles introduced by Hajac and collaborators among the projections classified.

1 Introduction

In the theory of noncommutative topology or geometry [5], a generally noncommutative C*-algebra $\mathcal{A}$ is viewed as the algebra $C(X_q)$ of continuous functions on a virtual spatial object X_q, called a quantum space. Many interesting examples of quantum spaces have been constructed with a topological or geometrical motivation, and analyzed in comparison with their classical counterpart. Different topological or geometrical viewpoints of the same object may give rise to different quantum versions of quantum spaces. For example, quantum odd-dimensional spheres and associated complex projective spaces have been introduced and studied by Soibelman, Vaksman, Meyer, and others [25, 12] as $\mathbb{S}_q^{2n+1}$ and $\mathbb{C}P_q^n$ via quantum universal enveloping algebra approach, and by Hajac and his collaborators including Baum, Kaygun, Matthes, Pask, Sims, Szymański, Zieliński, and others [2, 9, 8, 10] as $\mathbb{S}_H^{2n+1}$ and $\mathbb{P}^n(\mathcal{T})$ via a multi-pullback and Toeplitz algebra approach.

Recall that the concept of a vector bundle E over a compact space X can be reformulated in the noncommutative context as a finitely generated projective left module $\Gamma(E_q)$ over $C(X_q)$, viewed as the space of continuous cross-sections of some virtual noncommutative or quantum vector bundle E_q over X_q, as suggested by Swan's work [24]. Based on the strong connection approach to quantum principal bundles [7] for compact quantum groups [26, 27], Hajac and his collaborators introduced quantum line bundles L_k of degree k over $\mathbb{P}^n(\mathcal{T})$ as some rank-one projective modules realized as spectral subspaces $C\left(\mathbb{S}_H^{2n+1}\right)_k$ of $C\left(\mathbb{S}_H^{2n+1}\right)$ under a $U(1)$-action, and analyzed them via pairing of cyclic cohomology and K-theory [9, 10]. In particular, it was found that L_k is not stably free unless $k=0$, revealing some information about the K_0-group of $C(\mathbb{P}^n(\mathcal{T}))$. On the other hand, even for the most crucial case of $n=1$, the K_0-group was not fully determined despite the significant progress made in the 2003 paper [9].

Going beyond the K-theoretic study of C*-algebras that classifies finitely generated projective modules only up to stable isomorphism, some successes have been achieved in the study of cancellation problem, made popular by Rieffel [16, 17], that classifies finitely generated projective modules up to isomorphism, for some quantum algebras [17, 18, 20, 1, 14]. It is of interest and a natural question to classify finitely

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generated projective modules over $C(\mathbb{P}^n(\mathcal{T}))$ and identify the line bundles L_k among them, beside finding the K-groups of $C(\mathbb{P}^n(\mathcal{T}))$.

In this paper, we use the powerful groupoid approach to C*-algebras initiated by Renault [15] and popularized by Curto, Muhly, and Renault [6, 13] to realize $C(\mathbb{P}^n(\mathcal{T}))$ as a concrete groupoid C*-algebra [15]. Focusing on the quantum complex line $\mathbb{P}^1(\mathcal{T})$, we get the C*-algebra structure of $C(\mathbb{P}^1(\mathcal{T}))$ analyzed and its K-groups computed. Furthermore, we get the finitely generated left projective modules over $C(\mathbb{P}^1(\mathcal{T}))$ classified up to isomorphism by classifying the projections over $C(\mathbb{P}^1(\mathcal{T}))$, i.e. in $M_\infty(C(\mathbb{P}^1(\mathcal{T})))$, up to unitary equivalence, and explicitly identify the quantum line bundles L_k among the classified projections, showing that these modules L_k do exhaust all rank-one projections over $C(\mathbb{P}^1(\mathcal{T}))$. (After posting this work at ArXiv, a revised version of [10] appeared at arXiv containing a computation of K-groups of $C(\mathbb{P}^n(\mathcal{T}))$ for all n [11].)

2 Quantum projective spaces as groupoid C*-algebras

Taking the groupoid approach to C*-algebras initiated by Renault [15] and popularized by the work of Curto, Muhly, and Renault [6, 13], we give a description of the C*-algebras $C(\mathbb{S}_H^{2n-1})$ and $C(\mathbb{P}^{n-1}(\mathcal{T}))$ of [10] as some concrete groupoid C*-algebras. We refer to [15, 13] for the concepts and theory of groupoid C*-algebras used freely in the following discussion.

Let $\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n}$ with $n > 1$ be the transformation group groupoid $\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n$ restricted to the positive "cone" $\overline{\mathbb{Z}_\geq}^n$ where $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$ carries the standard topology and $\mathbb{Z}^n$ acts on $\overline{\mathbb{Z}}^n$ componentwise in the canonical way. From the invariant open subset $\mathbb{Z}_\geq^n$ of the unit space $\overline{\mathbb{Z}_\geq}^n$ of $\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n}$, we get the short exact sequence of C*-algebras

$$0 \to C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\mathbb{Z}_\geq^n}\right) \to C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n}\right) \to C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n \setminus \mathbb{Z}_\geq^n}\right) \to 0$$

and furthermore since the open invariant set $\mathbb{Z}_\geq^n$ is dense in the unit space, it induces a faithful representation π of $C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n}\right)$ on $\ell^2(\mathbb{Z}_\geq^n)$. From the groupoid isomorphism

$$\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n} \cong \times^n \left(\mathbb{Z} \ltimes \overline{\mathbb{Z}}\right)\big|_{\overline{\mathbb{Z}_\geq}}$$

and the C*-algebra isomorphism $C^*\left(\left(\mathbb{Z} \ltimes \overline{\mathbb{Z}}\right)\big|_{\overline{\mathbb{Z}_\geq}}\right) \cong \mathcal{T}$ for the Toeplitz C*-algebra $\mathcal{T}$ with $C^*\left(\left(\mathbb{Z} \ltimes \overline{\mathbb{Z}}\right)\big|_{\mathbb{Z}_\geq}\right) \cong \mathcal{K}\left(\ell^2(\mathbb{Z}_\geq)\right)$, we get

$$C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n}\right) \cong \otimes^n \mathcal{T}$$

with $C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\mathbb{Z}_\geq^n}\right) \cong \otimes^n \mathcal{K}\left(\ell^2(\mathbb{Z}_\geq)\right)$.

Since $C(\mathbb{S}_H^{2n-1}) \cong (\otimes^n \mathcal{T}) / (\otimes^n \mathcal{K})$ by (A.2) of [10], we have

$$C(\mathbb{S}_H^{2n-1}) \cong C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n}\right) \big/ C^*\left(\left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\mathbb{Z}_\geq^n}\right) \cong C^*(\mathfrak{G}_n)$$

realized as the groupoid C*-algebra of the concrete groupoid

$$\mathfrak{G}_n := \left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n\right)\Big|_{\overline{\mathbb{Z}_\geq}^n \setminus \mathbb{Z}_\geq^n}.$$

Next we note that the $U(1)$-action on $C\left(\mathbb{S}_H^{2n-1}\right) \cong C^*(\mathfrak{G}_n)$ considered in [10] is implemented by the multiplication operator

$$U_\zeta : f \in C_c(\mathfrak{G}_n) \mapsto h_\zeta f \in C_c(\mathfrak{G}_n)$$

for $\zeta \in U(1) \equiv \mathbb{T}$ where

$$h_\zeta : (m,p) \in \mathfrak{G}_n \subset \mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n \mapsto \zeta^{\Sigma m} \in \mathbb{T} \ \text{ with } \Sigma m := \sum_{i=1}^n m_i$$

is a groupoid character of $\mathfrak{G}_n$ and hence U_ζ is an automorphism of $C^*(\mathfrak{G}_n)$, since $h_\zeta h_{\zeta'} = h_{\zeta\zeta'}$ and $U_\zeta(w_i) = \zeta w_i$ for the generators w_i of $C\left(\mathbb{S}_H^{2n-1}\right)$ [10] identified with the characteristic function $\chi_{A_i} \in C_c(\mathfrak{G}_n)$ of

$$A_i := \left\{(e_i, p) : p \in \overline{\mathbb{Z}_\geq}^n \backslash \mathbb{Z}_\geq^n\right\} \subset \mathfrak{G}_n \subset \mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n.$$

The C*-algebra $C\left(\mathbb{P}^{n-1}(\mathcal{T})\right)$ of the quantum complex projective space studied in [8, 10] is then isomorphic to the $U(1)$-invariant C*-subalgebra $C^*(\mathfrak{G}_n)^{U(1)}$ of $C^*(\mathfrak{G}_n)$, which can be realized as the groupoid C*-algebra $C^*\left((\mathfrak{G}_n)_0\right)$ of the subgroupoid $(\mathfrak{G}_n)_0$ of $\mathfrak{G}_n$, where

$$(\mathfrak{G}_n)_k := \{(m,p) \in \mathfrak{G}_n : \Sigma m = k\}$$

for $k \in \mathbb{Z}$. Furthermore, $C^*(\mathfrak{G}_n)$ becomes a graded algebra $\oplus_{k\in\mathbb{Z}} \overline{C_c\left((\mathfrak{G}_n)_k\right)}$ with the component $\overline{C_c\left((\mathfrak{G}_n)_k\right)}$ being the quantum line bundle $C\left(\mathbb{S}_H^{2n-1}\right)_k$ [8, 10] of degree k over the quantum space $\mathbb{P}^{n-1}(\mathcal{T})$.

As shown in [10], the case of $\mathbb{P}^{n-1}(\mathcal{T})$ with $n = 2$ plays a crucially important role in the study of the quantum line bundle $C\left(\mathbb{S}_H^{2n-1}\right)_k$ in general, so we focus on the case of $\mathbb{P}^1(\mathcal{T})$ in the remaining part of this paper, while leaving the higher-dimensional cases to a subsequent paper.

3 K-groups of quantum projective line

In the case of $n = 2$, the groupoid $\mathcal{G} := \mathfrak{G}_2 \equiv \left(\mathbb{Z}^2 \ltimes \overline{\mathbb{Z}}^2\right)\Big|_{\overline{\mathbb{Z}_\geq}^2 \backslash \mathbb{Z}_\geq^2}$ has the unit space

$$\mathcal{G}^{(0)} = \overline{\mathbb{Z}_\geq}^2 \backslash \mathbb{Z}_\geq^2 = \left(\overline{\mathbb{Z}_\geq} \times \{\infty\}\right) \cup \left(\{\infty\} \times \overline{\mathbb{Z}_\geq}\right)$$

and consists of points (m,l,p,q) with $(m,l) \in \mathbb{Z}^2$ and $(p,q) \in \mathcal{G}^{(0)}$ such that $(m+p, l+q) \in \mathcal{G}^{(0)}$ where $m + \infty = \infty$ for any $m \in \mathbb{Z}$ is understood.

The previous discussion realizes the C*-algebra $C\left(\mathbb{P}^1(\mathcal{T})\right)$ of the quantum projective line $\mathbb{P}^1(\mathcal{T})$ as a groupoid C*-algebra $C^*(\mathcal{G}_0)$ where the subgroupoid

$$\mathcal{G}_0 := \left\{(n, -n, p, q) : n \in \mathbb{Z} \text{ such that } (p,q), (p+n, q-n) \in \mathcal{G}^{(0)}\right\} \subset \mathcal{G}$$

shares the same unit space $(\mathcal{G}_0)^{(0)} = \mathcal{G}^{(0)}$ with $\mathcal{G}$.

Note that the open dense invariant subset $U := (\mathbb{Z}_\geq \times \{\infty\}) \sqcup (\{\infty\} \times \mathbb{Z}_\geq)$ of $\mathcal{G}^{(0)}$ consists of two disjoint free orbits $\mathbb{Z}_\geq \times \{\infty\}$ and $\{\infty\} \times \mathbb{Z}_\geq$ of $\mathcal{G}_0$, from which we get a faithful representation π of $C^*(\mathcal{G}_0) \equiv C\left(\mathbb{P}^1(\mathcal{T})\right)$ on the Hilbert space

$$\mathcal{H} := \ell^2(U) \cong \ell^2(\mathbb{Z}_\geq) \oplus \ell^2(\mathbb{Z}_\geq)$$

such that $\pi\left(\delta_{(n,-n,p,q)}\right)$ for any $(n,-n,p,q) \in \mathcal{G}_0$ with $(p,q) \in U$ is the partial isometry sending $\delta_{(p,q)} \in \ell^2(U)$ to $\delta_{(p+n,q-n)} \in \ell^2(U)$ and all other $\delta_{(p',q')} \in \ell^2(U)$ to 0.

The open subgroupoid

$$\mathcal{G}_0|_U = \{(n,-n,p,\infty), (n,-n,\infty,q) : n \in \mathbb{Z} \text{ such that } p, q, p+n, q-n \in \mathbb{Z}_\geq\}$$

of $\mathcal{G}_0$ is isomorphic to the disjoint union $\mathfrak{K}_+ \sqcup \mathfrak{K}_-$ of two copies of the groupoid $\mathfrak{K} := (\mathbb{Z} \ltimes \mathbb{Z})|_{\mathbb{Z}_\geq}$ under the map $(n, -n, p, \infty) \mapsto (n, p) \in \mathfrak{K}_+$ and $(n, -n, \infty, q) \mapsto (-n, q) \in \mathfrak{K}_-$. Thus

$$C^* (\mathcal{G}_0|_U) \cong C^* (\mathfrak{K}_+) \oplus C^* (\mathfrak{K}_-) \cong \mathcal{K} \left(\ell^2 (\mathbb{Z}_\geq)\right) \oplus \mathcal{K} \left(\ell^2 (\mathbb{Z}_\geq)\right).$$

With $\mathcal{G}^{(0)} \backslash U = \{(\infty, \infty)\}$ and $\mathcal{G}_0|_{\{(\infty,\infty)\}} = \{(n, -n, \infty, \infty) : n \in \mathbb{Z}\}$ isomorphic to the group $\mathbb{Z}$, we have the short exact sequence

$$0 \to C^* (\mathcal{G}_0|_U) \cong \mathcal{K} \oplus \mathcal{K} \xrightarrow{\iota} C^* (\mathcal{G}_0) \xrightarrow{\sigma} C^* \left(\mathcal{G}_0|_{\{(\infty,\infty)\}}\right) \cong C (\mathbb{T}) \to 0,$$

where $C^* (\mathcal{G}_0|_U) \cong \mathcal{K} \oplus \mathcal{K}$ under the representation π with $\mathcal{K} \equiv \mathcal{K} \left(\ell^2 (\mathbb{Z}_\geq)\right)$, and $\delta_{(1,-1,\infty,\infty)} \in C^* \left(\mathcal{G}_0|_{\{(\infty,\infty)\}}\right)$ is identified with

$$z := \mathrm{id}_\mathbb{T} \in C (\mathbb{T}).$$

In the induced 6-term exact sequence

$$\begin{array}{ccccccc}
\mathbb{Z}[e_{11}] \oplus \mathbb{Z}[e_{11}] \cong & K_0 (\mathcal{K} \oplus \mathcal{K}) & \xrightarrow{K_0(\iota)} & K_0 (C^* (\mathcal{G}_0)) & \xrightarrow{K_0(\sigma)} & K_0 (C (\mathbb{T})) & \cong \mathbb{Z} \\
 & \uparrow \eta & & & & \downarrow \varepsilon & \\
\mathbb{Z}[z] \cong & K_1 (C (\mathbb{T})) & \xleftarrow{K_1(\sigma)} & K_1 (C^* (\mathcal{G}_0)) & \xleftarrow{K_1(\iota)} & K_1 (\mathcal{K} \oplus \mathcal{K}) & = 0,
\end{array}$$

the homomorphism $K_0 (\sigma)$ is clearly surjective, and we claim that the index homomorphism η sends $[z]$ to $(-[e_{11}]) \oplus [e_{11}]$, where e_{11} is the standard matrix unit and $z \equiv \mathrm{id}_\mathbb{T} \in GL_1 (C (\mathbb{T}))$. Indeed z lifts via σ to the characteristic function $\chi_W \in C^* (\mathcal{G}_0)$ of the set

$$W := \{(1, -1, p, \infty) : p \geq 0\} \cup \{(1, -1, \infty, q) : q \geq 1\},$$

and $\pi (\chi_W) = \mathcal{S} \oplus \mathcal{S}^*$ a partial isometry with kernel projection $0 \oplus e_{11}$ and cokernel projection $e_{11} \oplus 0$, where $\mathcal{S}$ is the (forward) unilateral shift. Hence

$$\eta ([z]) = [0 \oplus e_{11}] - [e_{11} \oplus 0] \in K_0 (\mathcal{K} \oplus \mathcal{K}).$$

(It is understood that the index homomorphism η used here may be different by a $\pm$-sign from the one used by other authors.)

Thus we get $K_0 (\iota) ([0 \oplus e_{11}] - [e_{11} \oplus 0]) = 0$ in $K_0 (C^* (\mathcal{G}_0))$, and hence $[e_{11} \oplus 0] = [0 \oplus e_{11}]$ in $K_0 (C^* (\mathcal{G}_0))$. A simple diagram chase concludes that $K_0 (C^* (\mathcal{G}_0)) \cong \mathbb{Z} [e_{11} \oplus 0] \oplus \mathbb{Z} \left[\tilde{I}\right]$ for the identity element $\tilde{I}$ of $C^* (\mathcal{G}_0)$, while $K_1 (C^* (\mathcal{G}_0)) = 0$. Furthermore

$$\begin{aligned} K_0 (\iota) : m [e_{11}] \oplus l [e_{11}] \in K_0 (\mathcal{K} \oplus \mathcal{K}) \cong \mathbb{Z} \oplus \mathbb{Z} \\ \mapsto (m + l) [e_{11} \oplus 0] \oplus 0 \left[\tilde{I}\right] \in K_0 (C^* (\mathcal{G}_0)) \cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

We summarize as follows.

Theorem 1. *For the quantum complex projective space $\mathbb{P}^1 (\mathcal{T})$, there is a short exact sequence of C^*-algebras decomposing its algebra $C \left(\mathbb{P}^1 (\mathcal{T})\right)$ as*

$$0 \to \mathcal{K} \oplus \mathcal{K} \xrightarrow{\iota} C \left(\mathbb{P}^1 (\mathcal{T})\right) \xrightarrow{\sigma} C (\mathbb{T}) \to 0,$$

and its K-groups coincide with those of its classical counterpart, i.e.

$$K_0 \left(C \left(\mathbb{P}^1 (\mathcal{T})\right)\right) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad K_1 \left(C \left(\mathbb{P}^1 (\mathcal{T})\right)\right) = 0.$$

4 Classification of projections over quantum projective line

In the following, we denote by $M_\infty(\mathcal{A})$ the direct limit (or the union as sets) of the increasing sequence of matrix algebras $M_n(\mathcal{A})$ over $\mathcal{A}$ with the canonical inclusion $M_n(\mathcal{A}) \subset M_{n+1}(\mathcal{A})$ identifying $x \in M_n(\mathcal{A})$ with $x \boxplus 0 \in M_{n+1}(\mathcal{A})$ for any algebra $\mathcal{A}$, where $\boxplus$ denotes the standard diagonal concatenation of two matrices. So the size of an element in $M_\infty(\mathcal{A})$ can be taken arbitrarily large. We also use $U_\infty(\mathcal{A})$ to denote the direct limit of the unitary groups $U_n(\mathcal{A}) \subset M_n(\mathcal{A})$ for a unital C*-algebra $\mathcal{A}$ with $U_n(\mathcal{A})$ embedded in $U_{n+1}(\mathcal{A})$ by identifying $x \in U_n(\mathcal{A})$ with $x \boxplus 1 \in U_{n+1}(\mathcal{A})$.

Before proceeding with the classification problem, we briefly recall the relation between projections over a C*-algebra $\mathcal{A}$ and finitely generated left projective modules over $\mathcal{A}$, and between them and K-theory.

By a projection over a unital C*-algebra $\mathcal{A}$, we mean a self-adjoint idempotent in $M_\infty(\mathcal{A})$. Two projections $P, Q \in M_n(\mathcal{A})$ are called unitarily equivalent if there exists a unitary $U \in M_N(\mathcal{A})$ with $N \geq n$ such that $UPU^{-1} = Q$. Each projection $P \in M_n(\mathcal{A})$ over $\mathcal{A}$ defines a finitely generated left projective module $\mathcal{A}^n P$ over $\mathcal{A}$ where elements of $\mathcal{A}^n$ are viewed as row vectors. The mapping $P \mapsto \mathcal{A}^n P$ induces a bijective correspondence between the unitary equivalence classes of projections over $\mathcal{A}$ and the isomorphism classes of finitely generated left projective modules over $\mathcal{A}$ [3].

Two finitely generated projective left modules E, F over $\mathcal{A}$ are called stably isomorphic if they become isomorphic after being augmented by the same finitely generated free $\mathcal{A}$-module, i.e. $E \oplus \mathcal{A}^k \cong F \oplus \mathcal{A}^k$ for some $k \geq 0$. Correspondingly, two projections P and Q are called stably equivalent if $P \boxplus I_k$ and $Q \boxplus I_k$ are unitarily equivalent for some identity matrix I_k. The K_0-group $K_0(\mathcal{A})$ classifies projections over $\mathcal{A}$ up to stable equivalence. The classification of projections over a C*-algebra up to unitary equivalence, appearing as the cancellation problem, was popularized by Rieffel's pioneering work [16, 17] and is in general an interesting but difficult question.

The set of all unitary equivalence classes of projections over a C*-algebra $\mathcal{A}$ is an abelian monoid $\mathfrak{P}(\mathcal{A})$ with its binary operation provided by the diagonal sum $\boxplus$ of projections. The image of the canonical homomorphism from $\mathfrak{P}(\mathcal{A})$ into $K_0(\mathcal{A})$ is the so-called positive cone of $K_0(\mathcal{A})$.

In the following, we use $\tilde{I}$ to denote the multiplicative unit of the unital C*-algebra $(\mathcal{K} \oplus \mathcal{K})^+ \subset C^*(\mathcal{G}_0)$ where $\mathcal{A}^+$ denotes the unitization of $\mathcal{A}$, and $\tilde{I}_n$ to denote the identity matrix in $M_n\left((\mathcal{K} \oplus \mathcal{K})^+\right)$, while

$$P_m := \sum_{i=1}^{m} e_{ii} \in M_m(\mathbb{C}) \subset \mathcal{K}$$

denotes the standard $m \times m$ identity matrix in $M_m(\mathbb{C}) \subset \mathcal{K}$ for any integer $m \geq 0$ (with $M_0(\mathbb{C}) \equiv \{0\}$ and $P_0 \equiv 0$ understood). We also use the notation

$$P_{-m} := I - P_m \in \mathcal{K}^+$$

for integers $m > 0$, where I is the identity operator canonically contained in $\mathcal{K}^+$, and symbolically adopt the notation

$$P_{-0} \equiv I - P_0 = I \neq P_0.$$

Furthermore by abuse of notation, we take

$$P_{-m} \oplus P_{-l} := \tilde{I} - (P_m \oplus P_l) \in (\mathcal{K} \oplus \mathcal{K})^+$$

if $m, l \geq 0$. Note that $P_m \oplus P_l \notin (\mathcal{K} \oplus \mathcal{K})^+$ if m and l are of strictly opposite $\pm$-sings.

Let $\alpha \in M_\infty(C^*(\mathcal{G}_0))$ be a projection. Since projections in $M_\infty(C(\mathbb{T}))$ are classified up to unitary equivalence as the constant functions on $\mathbb{T}$ with an identity matrix $I_n \in M_n(\mathbb{C})$ as the value for some $n \in \mathbb{Z}_\geq$ (and hence $K_0(C(\mathbb{T})) = \mathbb{Z}$), α is unitarily equivalent over $C^*(\mathcal{G}_0)$ to some projection $\beta \in M_N(C^*(\mathcal{G}_0))$ with $\sigma(\beta) = I_n$ for some $n \geq 0$ and a suitably large size $N \geq n$. It is easy to see that n depends only on α, and we call n the rank of α.

So in the following, we concentrate on classifying projections α over $C^*(\mathcal{G}_0)$ with $\sigma(\alpha) = I_n$ and $\alpha \in M_N(C^*(\mathcal{G}_0))$ for some $N \geq n$.

Now since $\sigma\left(\alpha-\tilde{I}_n\right)=I_n-I_n=0$,

$$\alpha-\tilde{I}_n\in M_N\left(\mathcal{K}\oplus\mathcal{K}\right)\equiv M_N\left(\mathcal{K}\right)\oplus M_N\left(\mathcal{K}\right)$$

which can be approximated by elements in $M_N\left(M_k\left(\mathbb{C}\right)\right)\oplus M_N\left(M_k\left(\mathbb{C}\right)\right)$. So we can replace α by a unitarily equivalent projection $\tilde{I}_n+x$ for some x in

$$M_N\left(M_k\left(\mathbb{C}\right)\right)\oplus M_N\left(M_k\left(\mathbb{C}\right)\right)\subset M_N\left(\pi\left(C_c\left(\mathcal{G}_0|_U\right)\right)\right)\subset M_n\left(\left(\mathcal{K}\oplus\mathcal{K}\right)^+\right)$$

with a suitably large k. Let I_n' be the identity element of

$$M_n\left(M_k\left(\mathbb{C}\right)\right)\oplus M_n\left(M_k\left(\mathbb{C}\right)\right)\subset M_n\left(\left(\mathcal{K}\oplus\mathcal{K}\right)^+\right).$$

Then since $I_n'+x\in M_N\left(M_k\left(\mathbb{C}\right)\right)\oplus M_N\left(M_k\left(\mathbb{C}\right)\right)$ is unitarily equivalent over $M_k\left(\mathbb{C}\right)\oplus M_k\left(\mathbb{C}\right)$ to $P_m\oplus P_l$ for some $0\leq m,l\leq Nk$ where $P_m\oplus P_l$ is the identity element of

$$M_m\left(\mathbb{C}\right)\oplus M_l\left(\mathbb{C}\right)\subset M_{Nk}\left(\mathbb{C}\right)\oplus M_{Nk}\left(\mathbb{C}\right)\equiv M_N\left(M_k\left(\mathbb{C}\right)\right)\oplus M_N\left(M_k\left(\mathbb{C}\right)\right),$$

we have $\tilde{I}_n+x$ unitarily equivalent over $\left(M_k\left(\mathbb{C}\right)\oplus M_k\left(\mathbb{C}\right)\right)^+\subset\pi\left(C_c\left(\mathcal{G}_0|_U\right)\right)^+$ to

$$\left(\tilde{I}_n-I_n'\right)+\left(P_m\oplus P_l\right)\in M_N\left(\left(M_k\left(\mathbb{C}\right)\oplus M_k\left(\mathbb{C}\right)\right)^+\right)\subset M_N\left(\left(\mathcal{K}\oplus\mathcal{K}\right)^+\right)$$

by the canonical embedding of $U_\infty\left(\mathcal{A}\right)$ in $U_\infty\left(\mathcal{A}^+\right)$ for any unital C*-algebra $\mathcal{A}$. Note that $\left(\tilde{I}_n-I_n'\right)+\left(P_m\oplus P_l\right)$ can be expressed in the form

$$(*)\quad\left(P_{m_1}\oplus P_{l_1}\right)\boxplus\cdots\boxplus\left(P_{m_N}\oplus P_{l_N}\right)\in M_N\left(\left(M_k\left(\mathbb{C}\right)\oplus M_k\left(\mathbb{C}\right)\right)^+\right)$$

for some $m_i,l_i\in\mathbb{Z}$ with $\left|m_i\right|,\left|l_i\right|\leq k$, and since $\sigma\left(\alpha\right)=\sigma\left(\tilde{I}_n\right)=I_n$, we have $m_i,l_i\leq 0$ (viewing $P_{m_i}\oplus P_{l_1}$ as $\tilde{I}-\left(P_{|m_i|}\oplus P_{|l_1|}\right)$) for $i\leq n$ and $m_i,l_i\geq 0$ for $i>n$.

It remains to classify projections $\alpha\in M_N\left(C^*\left(\mathcal{G}_0\right)\right)$ of the form (*) up to unitary equivalence over $C^*\left(\mathcal{G}_0\right)$.

When the rank n is 0, we have $m_i,l_i\geq 0$ for all i in (*), and $P_{m_i},P_{l_i}\in M_k\left(\mathbb{C}\right)$. With P_{m_i},P_{l_i} viewed as elements in $M_{Nk}\left(\mathbb{C}\right)\supset M_k\left(\mathbb{C}\right)$, the projections $P_{m_1}\boxplus\cdots\boxplus P_{m_N}$ and $P_{l_1}\boxplus\cdots\boxplus P_{l_N}$ lying in $M_N\left(M_{Nk}\left(\mathbb{C}\right)\right)$ with ranks bounded by Nk are unitarily equivalent over $M_{Nk}\left(\mathbb{C}\right)$ to $P_m\boxplus 0\boxplus\cdots\boxplus 0$ and $P_l\boxplus 0\boxplus\cdots\boxplus 0$ in $M_N\left(M_{Nk}\left(\mathbb{C}\right)\right)$ respectively, where $m:=\sum_i m_i\leq Nk$ and $l:=\sum_i l_i\leq Nk$. Hence $\left(P_{m_1}\oplus P_{l_1}\right)\boxplus\cdots\boxplus\left(P_{m_N}\oplus P_{l_N}\right)$ is unitarily equivalent over $\pi\left(C_c\left(\mathcal{G}_0|_U\right)\right)^+$ to $P_m\oplus P_l\in M_1\left(\left(\mathcal{K}\oplus\mathcal{K}\right)^+\right)\equiv\left(\mathcal{K}\oplus\mathcal{K}\right)^+$.

On the other hand, if such projections $P_m\oplus P_l$ and $P_{m'}\oplus P_{l'}$ with $m,l,m',l'\in\mathbb{Z}_\geq$ are unitarily equivalent over $C^*\left(\mathcal{G}_0\right)\subset\mathcal{B}\left(\ell^2\left(\mathbb{Z}_\geq\right)\right)\oplus\mathcal{B}\left(\ell^2\left(\mathbb{Z}_\geq\right)\right)$, then their ranks must coincide, i.e. $m=m'$ and $l=l'$. Thus for the case of $n=0$, we get unitary equivalence classes of projections over $C^*\left(\mathcal{G}_0\right)$ classified by $(m,l)\in\mathbb{Z}_\geq\times\mathbb{Z}_\geq$ as $P_m\oplus P_l$.

When the rank n is strictly positive, we claim that the projections $\tilde{I}_n$, $\tilde{I}_n\boxplus\left(P_m\oplus P_0\right)$, and

$$\tilde{I}_{n-1}\boxplus\left(P_{-m}\oplus P_{-0}\right)\equiv\tilde{I}_{n-1}\boxplus\left(\tilde{I}-\left(P_m\oplus 0\right)\right)$$

with $m\in\mathbb{N}\equiv\mathbb{Z}_>$, give a complete list of unitary equivalence classes of projections α with $\sigma\left(\alpha\right)=I_n$.

First we observe that for $m,l,n\geq 0$.

$$\left[\tilde{I}_n\boxplus\left(P_m\oplus P_l\right)\right]=\left[\tilde{I}_n\right]+K_0\left(\iota\right)\left(m\left[e_{11}\right]\oplus l\left[e_{11}\right]\right)=\left(m+l\right)\left[e_{11}\oplus 0\right]\oplus n\left[\tilde{I}\right]$$

and

$$\left[\tilde{I}_n \boxplus (P_{-m} \oplus P_{-l})\right] = \left[\tilde{I}_n \boxplus \left(\tilde{I} - (P_m \oplus P_l)\right)\right] = \left[\tilde{I}_n\right] + \left[\tilde{I} - (P_m \oplus P_l)\right]$$
$$= \left[\tilde{I}_n\right] + \left[\tilde{I}\right] - [P_m \oplus P_l] = (n+1)\left[\tilde{I}\right] - K_0(\iota)\left(m\,[e_{11}] \oplus l\,[e_{11}]\right)$$
$$= -(m+l)\,[e_{11} \oplus 0] \oplus (n+1)\left[\tilde{I}\right]$$

in $K_0\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$. So the stable equivalence class over $C\left(\mathbb{P}^1(\mathcal{T})\right)$ of a projection of the form $\tilde{I}_n \boxplus (P_m \oplus P_l)$ with $n, ml \geq 0$ (so m, l are integers not of opposite $\pm$-signs) is determined exactly by $m + l \in \mathbb{Z}$ and its rank (n or $n+1$). In particular, for $n > 0$, the projections $\tilde{I}_n$, $\tilde{I}_n \boxplus (P_m \oplus P_0)$, and $\tilde{I}_{n-1} \boxplus (P_{-m} \oplus P_{-0})$ with $m > 0$ are mutually stably and hence unitarily inequivalent.

It remains to show that any projection α of the form (*) with $n > 0$ is unitarily equivalent to one of $\tilde{I}_n$, $\tilde{I}_n \boxplus (P_m \oplus P_0)$, and $\tilde{I}_{n-1} \boxplus (P_{-m} \oplus P_{-0})$ with $m \in \mathbb{N}$.

Recall that $\sigma(\alpha) = I_n$ implies that $m_i, l_i \leq 0$ for $i \leq n$ and $m_i, l_i \geq 0$ for $i > n$. Since $\mathcal{K} \oplus \mathcal{K} \subset C\left(\mathbb{P}^1(\mathcal{T})\right)$, using some (unitary) finite permutation matrices, we can convert α to a unitarily equivalent projection β of the form

$$\beta = \left(\boxplus^{n-1} \tilde{I}\right) \boxplus (P_{m''} \oplus P_{l''}) \boxplus (P_{m'} \oplus P_{l'}) \boxplus \left(\boxplus^{N-n-1} 0\right) \in M_N\left((\mathcal{K} \oplus \mathcal{K})^+\right)$$

with $m'' = \sum_{i=1}^{n} m_i \leq 0$, $l'' = \sum_{i=1}^{n} l_i \leq 0$, $m' = \sum_{i=n+1}^{N} m_i \geq 0$, $l' = \sum_{i=n+1}^{N} l_i \geq 0$, or for short

$$\beta = \tilde{I}_{n-1} \boxplus (P_{m''} \oplus P_{l''}) \boxplus (P_{m'} \oplus P_{l'}) \in M_{n+1}\left((\mathcal{K} \oplus \mathcal{K})^+\right),$$

by swapping the largest (finite) identity diagonal blocks in P_{m_i} and P_{l_i} for $i > n+1$ with suitable disjoint diagonal zero blocks of $P_{m_{n+1}}$ and $P_{l_{n+1}}$ respectively, and by swapping the largest (finite) diagonal zero blocks in P_{m_i} and P_{l_i} for $i < n$ with suitable disjoint diagonal identity blocks of P_{m_n} and P_{l_n} respectively. Here it is understood that m'' and l'' carry a negative sign and hence $P_{m''} \oplus P_{l''} = \tilde{I} - \left(P_{|m''|} \oplus P_{|l''|}\right)$.

By swapping a suitable (finite) diagonal zero block of $P_{l''}$ with a suitable (finite) identity block of $P_{l'}$, we get β unitarily equivalent to either

$$\tilde{I}_{n-1} \boxplus (P_{m''} \oplus P_{l''+l'}) \boxplus (P_{m'} \oplus 0)$$

if $l'' + l' < 0$, or to

$$\tilde{I}_{n-1} \boxplus (P_{m''} \oplus P_{-0}) \boxplus (P_{m'} \oplus P_{l'+l''})$$

if $l'' + l' \geq 0$.

With $\pi(\chi_W) = \mathcal{S} \oplus \mathcal{S}^*$ as discussed earlier, conjugating $\tilde{I}_{n-1} \boxplus (P_{m''} \oplus P_{l''+l'}) \boxplus (P_{m'} \oplus 0)$ or $\tilde{I}_{n-1} \boxplus (P_{m''} \oplus P_{-0}) \boxplus (P_{m'} \oplus P_{l'+l''})$ by the unitary

$$\tilde{I}_{n-1} \boxplus \begin{pmatrix} \pi(\chi_W)^{|l''+l'|} & \tilde{I} - \pi(\chi_W)^{|l''+l'|}\left(\pi(\chi_W)^*\right)^{|l''+l'|} \\ \tilde{I} - \left(\pi(\chi_W)^*\right)^{|l''+l'|}\pi(\chi_W)^{|l''+l'|} & \left(\pi(\chi_W)^*\right)^{|l''+l'|} \end{pmatrix} \in M_{n+1}\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$$

or its adjoint respectively converts each to the form

$$\gamma := \tilde{I}_{n-1} \boxplus (P_{-j} \oplus P_{-0}) \boxplus (P_k \oplus 0)$$

for some $j, k \geq 0$ (up to swap of finite diagonal blocks in the first $\oplus$-summand).

Finally by swapping a suitable (finite) diagonal zero block of P_{-j} with a suitable (finite) identity block of P_k, we get γ unitarily equivalent to either $\tilde{I}_{n-1} \boxplus (P_{k-j} \oplus P_{-0}) \boxplus 0$ if $k - j < 0$, or $\tilde{I}_n \boxplus (P_{k-j} \oplus 0)$ if $k - j \geq 0$. Thus α is unitarily equivalent over $C\left(\mathbb{P}^1(\mathcal{T})\right)$ to either $\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_{j-k} \oplus 0)\right)$ if $k - j < 0$, or $\tilde{I}_n \boxplus (P_{k-j} \oplus 0)$ if $k - j \geq 0$ as wanted.

Now we summarize what we have found.

Theorem 2. *The abelian monoid* $\mathfrak{P}\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$ *of unitary equivalence classes of projections over* $C\left(\mathbb{P}^1(\mathcal{T})\right)$ *consists of (the representatives)* $P_m \oplus P_l$, $\tilde{I}_n \boxplus (P_j \oplus 0)$, *and* $\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)$ *for* $m, l, j \in \mathbb{Z}_{\geq}$ *and* $n, k \in \mathbb{Z}_{>}$, *where* $\tilde{I}$ *is the identity of* $(\mathcal{K} \oplus \mathcal{K})^+ \subset C\left(\mathbb{P}^1(\mathcal{T})\right)$ *and* P_k *is the identity element of* $M_k(\mathbb{C}) \subset \mathcal{K}$, *with its binary operation* $\cdot$ *specified by*

$$\begin{cases} (P_m \oplus P_l) \cdot \left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) = \tilde{I}_n \boxplus (P_{m+l+j} \oplus 0), & \\ \left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) \cdot \left(\tilde{I}_{n'-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_{n+n'} \boxplus (P_{j-k} \oplus 0) & \text{if} \quad j \geq k, \\ \left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) \cdot \left(\tilde{I}_{n'-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_{n+n'-1} \boxplus \left(\tilde{I} - (P_{k-j} \oplus 0)\right) & \text{if} \quad j < k, \\ (P_m \oplus P_l) \cdot \left(\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_n \boxplus (P_{m+l-k} \oplus 0) & \text{if} \quad m+l \geq k, \\ (P_m \oplus P_l) \cdot \left(\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_{k-m-l} \oplus 0)\right) & \text{if} \quad m+l < k, \end{cases}$$

for representatives of different types and by adding up corresponding indices m, l, j, n, k involved *for representatives of the same type.*

Corollary 1. *The cancellation law holds for projections* α *over* $C\left(\mathbb{P}^1(\mathcal{T})\right)$ *of rank* $n \geq 1$ *where* n *is the rank of the projection* $\sigma(\alpha) \in M_\infty(C(\mathbb{T}))$ *at any point of* T, *but fails for projections* α *over* $C\left(\mathbb{P}^1(\mathcal{T})\right)$ *of rank* 0.

We also get the following details about the positive cone of $K_0\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$, extending the information provided by Corollary 3.4 of [9].

Corollary 2. *The positive cone of* $K_0\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right) \cong \mathbb{Z}[e_{11} \oplus 0] \oplus \mathbb{Z}\left[\tilde{I}\right]$ *is*

$$\left(\mathbb{Z}_{\geq}[e_{11} \oplus 0] \oplus 0\left[\tilde{I}\right]\right) \cup \left(\mathbb{Z}[e_{11} \oplus 0] \oplus \mathbb{Z}_{>}\left[\tilde{I}\right]\right).$$

The canonical homomorphism from the monoid $\mathfrak{P}\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$ *to* $K_0\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$ *sends*

$$\begin{cases} P_m \oplus P_l & \mapsto \quad (m+l)[e_{11} \oplus 0] \\ \tilde{I}_n \boxplus (P_j \oplus 0) & \mapsto \quad j[e_{11} \oplus 0] \oplus n\left[\tilde{I}\right] \\ \tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right) & \mapsto \quad -k[e_{11} \oplus 0] \oplus n\left[\tilde{I}\right] \end{cases}$$

for $m, l, j \in \mathbb{Z}_{\geq}$ *and* $n, k \in \mathbb{Z}_{>}$.

We briefly compare the quantum complex projective space $\mathbb{P}^1(\mathcal{T})$ with the Podleś quantum sphere $\mathbb{S}^2_{\mu c}$ for $\mu \in (-1, 1)$ and $c > 0$, using the groupoid approach.

By the description of the structure of $C\left(\mathbb{S}^2_{\mu c}\right)$ in [22], $C\left(\mathbb{S}^2_{\mu c}\right)$ can be realized as the groupoid C*-algebra $C^*(\mathcal{F})$ of the subgroupoid

$$\mathcal{F} := \left\{(n, n, p, q) : n \in \mathbb{Z} \text{ such that } \quad (p, q), (p+n, q+n) \in \mathcal{G}^{(0)}\right\}$$

of $\mathcal{G}$, sharing the same unit space $\mathcal{F}^{(0)} = \mathcal{G}^{(0)}$ with $\mathcal{G}$.

The open subgroupoid

$$\mathcal{F}|_U = \{(n, n, p, \infty), (n, n, \infty, q) : n \in \mathbb{Z} \text{ such that } p, q, p+n, q+n \in \mathbb{Z}_{\geq}\}$$

of $\mathcal{F}$ is isomorphic to the disjoint union $\mathfrak{K}_+ \sqcup \mathfrak{K}_-$ of two copies of the groupoid $\mathfrak{K} := (\mathbb{Z} \ltimes \mathbb{Z})|_{\mathbb{Z}_{\geq}}$ under the map $(n, n, p, \infty) \mapsto (n, p) \in \mathfrak{K}_+$ and $(n, n, \infty, q) \mapsto (n, q) \in \mathfrak{K}_-$. Thus

$$C^*(\mathcal{F}|_U) \cong C^*(\mathfrak{K}_+) \oplus C^*(\mathfrak{K}_-) \cong \mathcal{K}\left(\ell^2(\mathbb{Z}_{\geq})\right) \oplus \mathcal{K}\left(\ell^2(\mathbb{Z}_{\geq})\right).$$

With $\mathcal{F}^{(0)} \backslash U = \{(\infty, \infty)\}$ and $\mathcal{F}|_{\{(\infty,\infty)\}} = \{(n, n, \infty, \infty) : n \in \mathbb{Z}\}$ isomorphic to the group $\mathbb{Z}$, we get the short exact sequence

$$0 \to C^*\left(\mathcal{F}|_U\right) \cong \mathcal{K} \oplus \mathcal{K} \xrightarrow{\iota} C^*(\mathcal{F}) \cong C\left(\mathbb{S}^2_{\mu c}\right) \xrightarrow{\sigma} C^*\left(\mathcal{F}|_{\{(\infty,\infty)\}}\right) \cong C(\mathbb{T}) \to 0,$$

where $\delta_{(1,1,\infty,\infty)} \in C^*\left(\mathcal{F}|_{\{(\infty,\infty)\}}\right)$ is identified with $z := \mathrm{id}_{\mathbb{T}} \in C(\mathbb{T})$ under σ. In the induced 6-term exact sequence of K-groups, the index homomorphism $\eta : K_1(C(\mathbb{T})) \to K_0(\mathcal{K} \oplus \mathcal{K})$ sends $[z]$ to $(-[e_{11}]) \oplus (-[e_{11}])$, and hence $K_0(\iota)([e_{11} \oplus 0]) = -K_0(\iota)([0 \oplus e_{11}])$, leading to $K_0(\iota)([P_m \oplus P_l]) = (m-l)[e_{11} \oplus 0]$ in $K_0\left(C\left(\mathbb{S}^2_{\mu c}\right)\right) \cong \mathbb{Z}[e_{11} \oplus 0] \oplus \mathbb{Z}\left[\tilde{I}\right]$.

By the same kind of analysis carried out above for $C\left(\mathbb{P}^1(\mathcal{T})\right)$, we get the following results.

Theorem 3. *The abelian monoid $\mathfrak{P}\left(C\left(\mathbb{S}^2_{\mu c}\right)\right)$ of unitary equivalence classes of projections over $C\left(\mathbb{S}^2_{\mu c}\right)$ consists of (the representatives) $P_m \oplus P_l$, $\tilde{I}_n \boxplus (P_j \oplus 0)$, and $\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)$ (or equivalently, $\tilde{I}_n \boxplus (0 \oplus P_k)$) for $m, l, j \in \mathbb{Z}_{\geq}$ and $n, k \in \mathbb{Z}_{>}$, where $\tilde{I}$ is the identity of $(\mathcal{K} \oplus \mathcal{K})^+ \subset C\left(\mathbb{S}^2_{\mu c}\right)$ and P_k is the identity of $M_k(\mathbb{C}) \subset \mathcal{K}$, with its binary operation $\cdot$ specified by*

$$\begin{cases}
(P_m \oplus P_l) \cdot \left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) = \tilde{I}_n \boxplus (P_{m+j-l} \oplus 0) & \text{if} \quad m+j \geq l, \\
(P_m \oplus P_l) \cdot \left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) = \tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_{l-m-j} \oplus 0)\right) & \text{if} \quad m+j < l, \\
\left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) \cdot \left(\tilde{I}_{n'-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_{n+n'} \boxplus (P_{j-k} \oplus 0) & \text{if} \quad j \geq k, \\
\left(\tilde{I}_n \boxplus (P_j \oplus 0)\right) \cdot \left(\tilde{I}_{n'-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_{n+n'-1} \boxplus \left(\tilde{I} - (P_{k-j} \oplus 0)\right) & \text{if} \quad j < k, \\
(P_m \oplus P_l) \cdot \left(\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_n \boxplus (P_{m-k-l} \oplus 0) & \text{if} \quad m \geq k+l, \\
(P_m \oplus P_l) \cdot \left(\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right)\right) = \tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_{k+l-m} \oplus 0)\right) & \text{if} \quad m < k+l,
\end{cases}$$

for representatives of different types and by adding up corresponding indices m, l, j, n, k involved *for representatives of the same type.*

Corollary 3. *The cancellation law holds for projections α over $C\left(\mathbb{S}^2_{\mu c}\right)$ of rank $n \geq 1$ where n is the rank of the projection $\sigma(\alpha) \in M_\infty(C(\mathbb{T}))$ at any point of $\mathbb{T}$, but fails for projections α over $C\left(C\left(\mathbb{S}^2_{\mu c}\right)\right)$ of rank* 0.

The following details about the positive cone of $K_0\left(C\left(\mathbb{S}^2_{\mu c}\right)\right)$ extend the information provided by Corollary 4.4 of [9].

Corollary 4. *The positive cone of $K_0\left(C\left(\mathbb{S}^2_{\mu c}\right)\right) \cong \mathbb{Z}[e_{11} \oplus 0] \oplus \mathbb{Z}\left[\tilde{I}\right]$ is*

$$\left(\mathbb{Z}[e_{11} \oplus 0] \oplus 0\left[\tilde{I}\right]\right) \cup \left(\mathbb{Z}[e_{11} \oplus 0] \oplus \mathbb{Z}_{>}\left[\tilde{I}\right]\right).$$

The canonical homomorphism from the monoid $\mathfrak{P}\left(C\left(\mathbb{S}^2_{\mu c}\right)\right)$ to $K_0\left(C\left(\mathbb{S}^2_{\mu c}\right)\right)$ sends

$$\begin{cases}
P_m \oplus P_l & \mapsto \quad (m-l)[e_{11} \oplus 0] \\
\tilde{I}_n \boxplus (P_j \oplus 0) & \mapsto \quad j[e_{11} \oplus 0] \oplus n\left[\tilde{I}\right] \\
\tilde{I}_{n-1} \boxplus \left(\tilde{I} - (P_k \oplus 0)\right) & \mapsto \quad -k[e_{11} \oplus 0] \oplus n\left[\tilde{I}\right]
\end{cases}$$

for $m, l, j \in \mathbb{Z}_{\geq}$ and $n, k \in \mathbb{Z}_{>}$.

Comparing the above results, we see that quantum complex projective lines $\mathbb{P}^1(\mathcal{T})$ and $\mathbb{S}^2_{\mu c}$ are distinguished apart by the monoid structures of $\mathfrak{P}\left(C\left(\mathbb{P}^1(\mathcal{T})\right)\right)$ and $\mathfrak{P}\left(C\left(\mathbb{S}^2_{\mu c}\right)\right)$, and also by the positive cone of their K_0-groups.

5 Line bundles over quantum projective line

In this section, we identify the quantum line bundles $L_k := C\left(\mathbb{S}^3_H\right)_k$ of degree k over $C\left(\mathbb{P}^1(\mathcal{T})\right)$ with a concrete (unitary equivalence class of) projection classified in the previous section.

To distinguish between ordinary function product and convolution product, we denote the groupoid C*-algebraic multiplication of elements in $C^*(\mathcal{G}) \supset C_c(\mathcal{G})$ by $*$, while omitting $*$ when the elements are presented as operators or when they are multiplied together pointwise as functions.

Recall from earlier section that $L_k = \overline{C_c(\mathcal{G}_k)} \subset C(\mathbb{S}^3_H)$ where

$$\mathcal{G}_k := \left\{(n+k, -n, p, q) : n \in \mathbb{Z} \text{ such that } \ (p,q), (p+n+k, q-n) \in \mathcal{G}^{(0)}\right\} \subset \mathcal{G},$$

and $L_0 = \overline{C_c(\mathcal{G}_0)} = C\left(\mathbb{P}^1(\mathcal{T})\right)$. Furthermore the groupoid C*-algebra $C^*(\mathcal{G}) \cong C(\mathbb{S}^3_H) = \overline{\oplus_{k\in\mathbb{Z}} L_k}$ is a $\mathbb{Z}$-graded algebra.

Let $k > 0$. We identify below separately L_k and L_{-k} with a representative of projections over $C\left(\mathbb{P}^1(\mathcal{T})\right)$ classified earlier.

The characteristic function $\chi_A \in C_c(\mathcal{G}_k) \subset C(\mathbb{S}^3_H)$ of the compact set

$$A := \left\{(k, 0, p, q) : (p,q), (p+k, q) \in \mathcal{G}^{(0)}\right\} \subset \mathcal{G}$$

is an isometry with $\chi_A^* * \chi_A = \chi_{\mathcal{G}^{(0)}} \equiv 1 \in C^*(\mathcal{G}_0)$ and

$$\chi_A * \chi_A^* = \chi_B = \tilde{I} - (P_k \oplus 0)$$

a projection in $C^*(\mathcal{G}_0) \equiv C\left(\mathbb{P}^1(\mathcal{T})\right)$for the set

$$B := \{(0, 0, p', q') \in \mathcal{G} : \ p' \geq k, \ q' \geq 0\}.$$

So with $\chi_B * \chi_A = \chi_A \in C_c(\mathcal{G}_k)$, we get a left $C^*(\mathcal{G}_0)$-module homomorphism

$$x \in C^*(\mathcal{G}_0) * \chi_B \mapsto x * \chi_A \in \overline{C_c(\mathcal{G}_k)} \equiv L_k$$

with well-defined inverse map

$$y \in \overline{C_c(\mathcal{G}_k)} \mapsto y * \chi_A^* = y * \chi_A^* * \chi_B \in C^*(\mathcal{G}_0) * \chi_B$$

since $\chi_A^* \in C_c(\mathcal{G}_{-k})$ and hence $C_c(\mathcal{G}_k) * \chi_A^* \subset C_c(\mathcal{G}_0)$. Now L_k being isomorphic to the left $C^*(\mathcal{G}_0)$-module $C^*(\mathcal{G}_0)\left(\tilde{I} - (P_k \oplus 0)\right)$ is identified with the rank-one projection $\tilde{I} - (P_k \oplus 0)$.

Next we show that in the left $C^*(\mathcal{G}_0)$-module decomposition

$$L_{-k} = L_{-k} * \chi_B \oplus L_{-k} * (1 - \chi_B) \equiv L_{-k}\left(\tilde{I} - (P_k \oplus 0)\right) \oplus L_{-k}(P_k \oplus 0)$$

by the projection χ_B, the first component $L_{-k} * \chi_B$ can be identified with the projection $\tilde{I}$ and the second component $L_{-k} * (1 - \chi_B)$ can be identified with the projection $P_k \oplus 0$, leading to the conclusion that L_{-k} can be identified with the projection $\tilde{I} \boxplus (P_k \oplus 0)$.

Indeed since $\chi_A * \chi_A^* = \chi_B$ and $\chi_A^* * \chi_A = 1 \equiv \chi_{\mathcal{G}^{(0)}}$ with $\chi_A^* \in C_c(\mathcal{G}_{-k})$, the map

$$x \in C^*(\mathcal{G}_0) \mapsto x * \chi_A^* = x * \chi_A^* * \chi_B \in \overline{C_c(\mathcal{G}_{-k})} * \chi_B \equiv L_{-k} * \chi_B$$

is a $C^*(\mathcal{G}_0)$-module isomorphism with inverse $y \mapsto y * \chi_A$ and hence $L_{-k} * \chi_B$ is identified with the projection $\tilde{I}$.

On the other hand, comparing

$$\begin{aligned} C_c(\mathcal{G}_{-k}) * (1 - \chi_B) &= C_c(\{(n-k, -n, p, \infty) : 0 \leq p < k \text{ and } p+n \geq k\}) \\ &= C_c(\{(n, -n-k, p, \infty) : 0 \leq p < k \text{ and } p+n \geq 0\}) \end{aligned}$$

and

$$C_c(\mathcal{G}_0) * (1 - \chi_B) = C_c(\{(n, -n, p, \infty) : 0 \leq p < k \text{ and } p+n \geq 0\})$$

where with the last coordinate being ∞, the second coordinate becomes irrelevant, we get a $(C_c(\mathcal{G}_0), *)$-module isomorphism

$$f \in C_c(\mathcal{G}_0) * (1 - \chi_B) \mapsto f \circ \tau \in C_c(\mathcal{G}_{-k}) * (1 - \chi_B)$$

where $\tau(n, -n-k, p, \infty) := (n, -n, p, \infty)$, which extends to a $C^*(\mathcal{G}_0)$-module isomorphism

$$C^*(\mathcal{G}_0) * (1 - \chi_B) \equiv C^*(\mathcal{G}_0)(P_k \oplus 0) \to \overline{C_c(\mathcal{G}_{-k})} * (1 - \chi_B) \equiv L_{-k} * (1 - \chi_B).$$

So the $C^*(\mathcal{G}_0)$-module $L_{-k} * (1 - \chi_B)$ is identified with the projection $P_k \oplus 0$.

We summarize as follows.

Theorem 4. *The quantum line bundle $L_k \equiv C(\mathbb{S}^3_H)_k$ of degree $k \in \mathbb{Z}$ over $C(\mathbb{P}^1(\mathcal{T}))$ is isomorphic to the finitely generated projective left module over $C(\mathbb{P}^1(\mathcal{T}))$ determined by the projection $\tilde{I} - (P_k \oplus 0)$ if $k \geq 0$, and the projection $\tilde{I} \boxplus (P_{-k} \oplus 0)$ if $k < 0$.*

Corollary 5. *The quantum line bundles $L_k \equiv C(\mathbb{S}^3_H)_k$ with $k \in \mathbb{Z}$ provide a complete list of mutually non-isomorphic rank-one finitely generated left projective modules over $C(\mathbb{P}^1(\mathcal{T}))$.*

It is interesting to note that in the case of quantum teardrops $WP_q(k,l)$, the quantum principal $U(1)$-bundles $\mathcal{L}(k)$ of degree k over $C(WP_q(k,l))$ introduced by Brzeziński and Fairfax [4] do not exhaust all rank-one finitely generated projective modules over $C(WP_q(k,l))$ by the result of [23].

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